MULTIPLE EQUILIBRIA
IN BODY-MASS

Rodrigo Cerda

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Multiple Equilibria in Body-mass

Rodrigo A. Cerda
Economics Department
Catholic University of Chile.

Address:
Economics Department, Catholic University of Chile,
Vicuña Mackenna 4860, Macul, Santiago, Chile
Tel: (562) 354-7101, Fax: (562) 553-2377,
e-mail: rcerda@faceapuc.cl.

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Abstract
This paper develops a simple model in which individuals rationally determine their body-mass by choosing food ingestion and time spent exercising. We show that multiple equilibria in body-mass might exist due to two forces with opposite effects. Firstly, an increase in body-mass has a negative impact on current utility and therefore, slows down body-mass accumulation. Secondly, an increase in body-mass has negative impact on the mortality rate, and thus on the individual’s discount factor. This effect is associated with an individual’s “myopic” behavior as more weight is given to current utility flow compared with future utility flows. As a result, the impact on the mortality rate accelerates the accumulation of body-mass throughout an increase in food ingestion and less time allocated to exercise. Thus, some individuals might be willing to ingest less food and spent more time exercising if they place more value on the negative impact of body-mass on their current utility, while others individuals might decide to accelerate body-mass if they face lower discounted future utility flows. A second conclusion relies on the stability of the different equilibria, which assures persistence in body-mass and explains why radical treatments might be required to modify an individual’s weight.
INTRODUCTION\(^1\)

The prevalence of eating and weight disorders is considerably high. At least a third of all Americans are now considered to be obese and sixty-percent are overweight. Also in the United States, five to ten million girls and women and one million boys and men are struggling with eating disorders including anorexia, bulimia or binge eating disorder (Crowther et al., 1992; Fairburn et al., 1993; Gordon, 1990; Hoek, 1995; Shisslak et al., 1995). In The UK, a study by the Royal College of Psychiatrists found about 60,000 people receiving some form of treatment for an eating disorder. Studies in the mid 1990s indicate the total number of people affected by an eating disorder in the UK to be in the region of 1.1 million.

The cost of treatment of eating disorders might be quite considerable. If the disease becomes acute and the person needs expensive medical monitoring and treatment, the cost of in-patient treatment can be $30,000 or more per month\(^2\). Further, many people will need repeated hospitalizations.

These episodes are very persistent. Seventy-seven percent of individuals with eating disorders report that the illness can last anywhere from one to fifteen years or even longer in some cases. Only fifty percent all people with this devastating disease report being cured\(^3\).

Dieting, which might be thought as another but less severe type of eating disorder, is a common phenomena among “normal weight” individuals. Ninety-one percent of

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\(^1\) This paper benefited from comments from Julio Elias, Gert Wagner and Felipe Zurita. Any remaining error is my own responsibility

\(^2\) Source: Eating Disorders Recovery Online.

\(^3\) Source: National Association of Anorexia Nervosa and Associated Disorders – ANAD
women surveyed on a college campus had attempted to control their weight through dieting while twenty-two percent dieted "often" or "always" (Kurth et al., 1995). Similarly a study by Smolak (1996) indicated that twenty-five percent of American men and forty-five percent of American women are on a diet on any given day. The same study shows that Americans spend over forty billion dollars on dieting and diet-related products each year. These diets are generally not effective. In fact, ninety-five percent of all dieters will regain their lost weight in one to five years (Grodstein, 1996).

Eating and weight disorders are complex conditions that arise from a combination of long-standing behavioral, emotional, psychological, interpersonal, and social factors. Nevertheless, they may also be rationalized in an economic content: individuals choose, by comparing costs and benefits, food ingestion and the amount of time they spend in exercising. Thus economic rationale may play a role in determining optimal body-mass (weight) of individuals. This paper will rely in a mathematical framework, based in economic rationale, which reproduce some of the observations concerning eating and weight disorders.

There is a growing academic debate concerning obesity and weight disorders in the health economic literature -see Lakdawalla and Philipson (2002) Philipson and Posner (1999), Levy (2002). The paper by Levy is mathematical appealing and provides interesting insights concerning the individual’s rationality in determining her body-mass. It is fair to characterize the main conclusion of this study as follows: rational behavior is associated with overweightness on the individual’s long run body-mass and small deviations from this condition lead to explosive oscillations in body-mass, which converge to states of severe underweightness. However, the result concerning the long run equilibrium of
overweightness seems to contradict the observation of individuals with severe low weight (anorexic) or normal weight. Further, the more striking result relies on the conclusion that overweight individual may converge to severe underweightness throughout explosive oscillations. In the dynamics, the variance of body-mass rises through time until convergence is acquired at severe low weight, e.g. an overweight individual might become fatter, then thinner, then even fatter, and so on until converging to severe underweightness. This result is striking because even though there might exist oscillations in body-mass, the evidence suggests large persistence on body-mass, e.g. deviations from the equilibrium weight -such as dieting- are not sustainable, converging back to the initial body-mass. Thus, while cycles in food ingestions and weight might exist, they are generally not explosive.

This paper will propose a framework of optimal control with lack of overall concavity in body-mass. The solution to the model have various characteristics. Firstly, there are multiple equilibria -rather than in a unique equilibrium related to overweightness and explosive oscillations as in Levy (2002)- in which rational decisions may lead to states of obesity, anorexia or normal weight. Secondly, equilibria with larger body-mass are associated with higher food ingestion and less time spent in exercising. Thirdly, those states coexist for the same individual and the transition to them occur through a saddle-path or through converging oscillations. Convergence to any of the equilibria depends on initial body-mass. As initial body-mass is exogenous to the individual’s problem, this variable might resume the influence of non-economic factors such as genetics, psychological or social factors. Thus, those variables determine the individual’s long run body-mass while the path to the equilibrium depends on rational decisions. Fourthly, those equilibria are stable, which assures persistence in body-mass and discards explosive oscillations.
The paper is developed as follows. Section I describe the economic environment, while section II characterizes the equilibria of the model. Finally, section III discusses the results.

I. THE ECONOMIC ENVIRONMENT

We will focus in a continuous time problem in which individuals have perfect foresight. We describe next each individual’s problem.

**Endowment:** The individual is endowed with (1) some initial body-mass, \( W_0 > 0 \), and (2) a unit of time that can be used to exercise or can be supplied to the labor market. We will denote time used to exercise as \( s_t \in \mathbb{R}_+, s_t \leq 1 \). Thus labor income is \( y_t (1 - s_t) \) where \( y_t \) is the wage rate per unit of time. To simplify the problem, we will assume the wage rate being constant through time, e.g. \( y_t = y \).

**Preferences:** Labor income is used to purchase food, \( F_t \in \mathbb{R}_+ \), which provides utility. The individual’s instantaneous utility function will be assumed to be a separable function between food ingestion and body-mass as in \( u(F_t, W_t) \), with properties \( u_F > 0, u_W < 0, u_{FF} < 0, u_{WW} < 0 \). Those assumptions assures concavity of the current utility flow on food ingestion and body-mass. We also assume that \( u \in C^2 \) and satisfies the following conditions, \( \lim_{F \to 0} u_F = \infty, \lim_{F \to \infty} u_F = 0, \lim_{W \to 0} u_W = 0, \lim_{W \to \infty} u_W = -\infty \). Individual’s survival rate is \( \exp(-\int_0^t \lambda(W) \, d\varepsilon) \), where the instantaneous death rate, \( \lambda(W_t) \), is increasing.
in body-mass, e.g., $\lambda_W > 0$. Hence, expected utility is defined as
\[
J(F_t, W_t) = u(F_t, W_t)e^{-\rho \int_0^\infty \lambda(W_s)ds}, \quad \text{where } \rho > 0 \text{ is a constant discount factor.}
\]

**Constraints:** The evolution of body-mass will depend positively on food ingestion and negatively in exercise. Indeed, exercise will affect body-mass by raising metabolism and thus increasing the body mass’ depreciation rate, $\delta(s_t)$, where $\delta_s > 0, \delta_{ss} < 0$. Also, food ingestion will be mapped to body-mass by a function $g(F_t)$, where $g_F > 0, g_{FF} < 0$. These functions satisfy Inada conditions, namely $\lim_{s \to 0} \delta_s = \infty, \lim_{s \to \infty} \delta_s = 0$, $\lim_{F \to 0} g_F = \infty, \lim_{F \to \infty} g_F = 0$. Hence the constraints of the problem are:

\[\dot{W} = \delta(s_t)W + g(F_t) \quad (1)\]
\[F_t = y(1 - s_t) \quad (2)\]

**Individual’s optimization problem:** Given the initial body-mass stock, $W_0$, the individual maximizes the discounted sum of instantaneous utility,
\[
\int_0^\infty u(F_t, W_t)e^{-\rho \int_0^t \lambda(W_s)ds} dt \quad \text{by choosing the set of paths } \{s_t, F_t, W_t\}_{t=0}^\infty \text{ subject to (1)-(2), and the constraints } F_t \geq 0, s_t \in [0,1], \text{ where } \rho > 0 \text{ is a constant discount factor. Under the assumption on the utility and depreciation functions, no corner solutions will occur.
Necessary condition for optimality: We define the current value Hamiltonian,

\[ H:\mathbb{R}^3 \rightarrow \mathbb{R}, \quad \text{as in} \quad H(F_t, W_t, \mu_t) = u(F_t, W_t)e^{-\rho t} \int_{s}^{t} \lambda(W_s)ds + \mu_t \left[ \delta \left(1 - \frac{F_t}{y}\right) W_t + g(F_t) \right], \]

where \( \mu_t \) is the shadow price (costate variable) of food ingestion and (1)-(2) have been combined in a unique constraint. Maximizing \( H \) with respect to \( F_t \) yields the usual equality between marginal utility and marginal cost of food ingestion:

\[ u_t e^{-\rho t} \int_{s}^{t} \lambda(W_s)ds = \mu_t \left[ -\delta \frac{W_t}{y} - g_W \right] \quad (3) \]

Since \( \delta, g_W > 0, \mu_t < 0 \). Let the present value shadow price be

\[ \psi_t = \frac{\mu_t}{\exp \left(-\rho t - \int_{s}^{t} \lambda(W_s)ds\right)} \],

then equation (3) yields an implicit demand function for food ingestion, \( F(\psi_t, W_t) \). Moreover, simply differentiation indicates \( F_\psi > 0, F_W < 0 \). An exogenous increase in the shadow price of body-mass, which measures its marginal contribution, will promote food ingestion and body-mass accumulation while an exogenous increase in body-mass will decrease food ingestion to offset the negative impact of the exogenous increase of body-mass. Using these results we define the present value maximized Hamiltonian, \( H^0 \) (where the time index was suppressed to simplify the notation)

**Definition.** The maximized present value Hamiltonian, \( H^0 : \mathbb{R}^2 \rightarrow \mathbb{R}, \) is given by:

\[ H^0 (W, \psi) = \max_F H(F, W, \mu) = u(F(W, \psi), W) + \psi \left[ -\delta \left(1 - \frac{F(W, \psi)}{y}\right) \right] W + g(F(W, \psi)) \]  

(4)
This is a useful property because it allows us to describe the evolution of the variables of interest \( (F,s,W) \) in the space \( (W,\psi) \) by the following set of stationary differential equations:

\[
\dot{H}_W^0 = \dot{W} = -\delta \left( 1 - \frac{F(W,\psi)}{y} \right) W + g(F(W,\psi)) \tag{5}
\]

\[
\dot{\psi} = -H_W^0 + [\rho + \lambda(W)]\psi = -\left[ u_W + \psi \delta \left( 1 - \frac{F(W,\psi)}{y} \right) \right] + [\rho + \lambda(W)]\psi \tag{6}
\]

and \( W_0 > 0 \).

Equation (6) can be interpreted as an arbitrage condition between current and future body-mass. To do so, note that equation (6) is an asset-pricing formula: \( \psi \) is the shadow price of body-mass in terms of current utility, \( H_W^0 \) is the return received by the individual (marginal contribution of body-mass to current utility), \( \dot{\psi} \) is the capital gain (change in the price of body-mass) and \( [\rho + \lambda(W)] \) is the return of increasing body-mass in the future, rather than today. The variable \( [\rho + \lambda(W)] \) will be denoted as “modified” discount factor, as it consider the impact of mortality rate on the discounting of future flow of utility. Note that \( H_W^0 \) is composed by two elements: the direct impact on current utility of an additional unit of body mass and the impact on body-mass depreciation. As a matter of notation, let define \( \tilde{H}_W^0 = \left[ u_W + \psi \delta \left( 1 - \frac{F(W,\psi)}{y} \right) \right] - [\rho + \lambda(W)]\psi \). Thus, the variable \( \tilde{H}_W^0 \) represents the net rate of return of body-mass, composed by the impact on current utility and the effect on the discount rate of future utility flows.
II. CHARACTERIZATION OF THE EQUILIBRIA

To describe the equilibrium, we focus on the phase diagram determined by the loci (5) and (6). The slopes of the loci $\psi = 0, \dot{W} = 0$ are:

$$
\frac{\partial \psi}{\partial W} \bigg|_{\dot{W}=0} = -\frac{H_{\psi W}^0}{H_{\psi \psi}^0}
$$

$$
\frac{\partial \psi}{\partial W} \bigg|_{\psi=0} = -\frac{\tilde{H}_{\psi W}^0}{\tilde{H}_{\psi \psi}^0}
$$

Simply differentiation shows that $H_{\psi W}^0 > 0, H_{\psi \psi}^0, \tilde{H}_{\psi \psi}^0 < 0$. Thus $\frac{\partial \psi}{\partial W} \bigg|_{\dot{W}=0} > 0$.

The sign of $\tilde{H}_{\psi W}^0$ depends on:

$$
\tilde{H}_{\psi W}^0 = u_{\psi W} - \lambda_{\psi W} \psi + \delta_{\psi} \frac{F_{\psi W}}{\psi}
$$

(7)

Note that $\tilde{H}_{\psi W}^0$ is the impact of a marginal increase of body-mass on its net rate of return. Since $u_{\psi W}, F_{\psi W}, \psi < 0, \delta_{\psi}, \lambda_{\psi W} > 0$, the sign of $\tilde{H}_{\psi W}^0$ is ambiguous.

Thus, the result indicates that the net rate of return on body-mass is not monotonically decreasing. The intuition on this result relies on the following factors. Firstly, the direct impact on current utility, $u_{\psi W}$, decreases the net rental rate from current body-mass, because individuals face larger negative effects on their marginal utility as body-mass raises. Secondly, there are other factors which affects the rental rate in the opposite direction. The modified discount factor raise through the impact on the mortality rate, providing less incentives to switch body-mass accumulation to the future. This is some type of “myopic” reaction as individuals care “less” about their future when body-mass
raises. In addition, the net rental rate might also be positively affected because at least part of the increase in body-mass can be depreciated. To proceed with the analysis, we will distinguish two cases: (1) the net rental rate being globally decreasing, e.g. \( \tilde{H}_{WW}^0 \leq 0, \forall W \) and (2) the net rental rate not being globally decreasing, e.g. ranges of body-mass with \( \tilde{H}_{WW}^0 \leq 0 \) while others with the opposite condition.

When \( \tilde{H}_{WW}^0 \leq 0, \forall W \), both loci cross once in the space \((W, \psi)\) and there exists a unique equilibrium. We may study the local stability of the system (5)-(6) by the use of a local approximation around the unique steady-state, \((W^{ss}, \psi^{ss})\) as in:

\[
\begin{bmatrix}
\dot{W} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
-H_{\psi W}^{0,ss} & H_{\psi \psi}^{0,ss} \\
-\tilde{H}_{WW}^{0,ss} & -\tilde{H}_{W \psi}^{0,ss}
\end{bmatrix}\begin{bmatrix}
W - W^{ss} \\
\psi - \psi^{ss}
\end{bmatrix}
\]

Where SS indicates evaluation at the steady state. Note that the determinant of (8) is negative with two real-valued eigenvalues. In fact, the eigenvalues are:

\[\gamma = \frac{-b \pm \sqrt{b^2 - 4c}}{2}\]

Where \( b = \tilde{H}_{\psi W}^0 - H_{\psi W}^0 = \rho + \delta + \lambda(W_i) + F_{\psi} \left[ \frac{\delta}{y} W + g_F \right] > 0 \), \( c = \tilde{H}_{WW}^0 H_{\psi \psi}^0 - \tilde{H}_{W \psi}^0 H_{\psi W}^0 \).

and c is the determinant of (8). Since \( c < 0 \), the eigenvalues are real-valued with opposite signs, thus there is saddle-path stability. To completely characterize the phase diagram note that to the right of the locus \( \dot{\psi} = 0 \), the shadow price raises since \( \frac{\partial \psi}{\partial W} = -\tilde{H}_{WW}^0 > 0 \) while above the locus \( \dot{W} = 0 \), body-mass increases since \( \frac{\partial W}{\partial \psi} = \left( \frac{\delta}{y} + g_F \right) F_{\psi} > 0 \). Figure 1 illustrates this case. As shown in the figure, there is a unique equilibrium. This equilibrium
is stable with no oscillations. Thus, any deviation from the long run equilibrium is smoothly eliminated.

Figure 1: Phase diagram, $\tilde{H}^0_{WW}$ globally decreasing

Next consider the case when $\tilde{H}^0_{WW}$ is not globally decreasing, e.g. there might exists ranges of values of body-mass in which $\tilde{H}^0_{WW} > 0$ while there are other ranges of values of $W$ in which $\tilde{H}^0_{WW} < 0$. It follows that on the former case $\frac{\partial \psi}{\partial W} < 0$ while the opposite holds on the later case. This, in turn, implies that both loci might cross more than once and,
hence, there might be multiple stationary points. Some of them associated with \( \tilde{H}_{WW}^0 < 0 \) while others with \( \tilde{H}_{WW}^0 > 0 \).

From above we know that in any stationary point in which \( \tilde{H}_{WW}^0 < 0 \), there is saddle path stability, as the eigenvalues have opposite signs. Conversely, consider the case of an equilibrium associated with \( \tilde{H}_{WW}^0 > 0 \). There are two cases to consider. Firstly, suppose \( \theta \geq \tilde{H}_{WW}^0 > 0 \) where \( \theta > 0 \) is chosen such that \( c < 0 \) and thus \( b^2 - 4c \geq 0 \). Secondly, suppose \( \tilde{H}_{WW}^0 > \theta \) such that \( c > 0 \) and \( b^2 - 4c < 0 \). The first case has similar implications to an equilibrium in which \( \tilde{H}_{WW}^0 < 0 \). However, the existence of an equilibrium in which \( \tilde{H}_{WW}^0 > 0 \) requires the locus \( \dot{\psi} = 0 \) to cross from below the locus \( \dot{W} = 0 \), and thus \[ \frac{\partial \dot{\psi}}{\partial W|_{\dot{\psi}=0}} > \frac{\partial \psi}{\partial W|_{\psi=0}} > 0. \] This condition is satisfied only when \( c > 0 \). Therefore the first case might be discarded in an equilibrium in which \( \tilde{H}_{WW}^0 > 0 \). We may conclude that in of these type of equilibria, the associated eigenvalues are complex with negative real parts which implies the stationary point being stable and oscillating. Figure 2 illustrates the result with three equilibria. There are three equilibria because there is a unique range of value in which \( \tilde{H}_{WW}^0 > 0 \). All of the stationary points are stable and only one is oscillating.

Finally, note that in a stationary equilibrium, it holds:

\[
0 = \dot{W} = -\delta \left( 1 - \frac{F}{y} \right) W^{ss} + g(F)
\]

\[\Rightarrow W^{ss} = \frac{g(F)}{\delta \left( 1 - \frac{F}{y} \right)}\]
Since $g_e, \delta_s > 0$ an increase in the stationary body-mass requires larger food ingestion. Also from (2), larger food ingestion implies less time spent exercising. Thus, we may conclude that an equilibrium with larger body-mass is associated with larger food ingestion and less exercise.

Figure 2: Phase diagram, $\tilde{H}_{WW}^0$ not globally decreasing, complex eigenvalues
III. Discussion

Figure 2 illustrates the existence of three equilibria. Those equilibria can be ranked as a function of their associated body-mass. If point B is associated with a “normal” weight individual, point A would represent an equilibrium state of underweightness while point C would represent an equilibrium state of overweightness.

The persistence of these equilibria is assured by their property of stability. This pattern might explain why individuals gain weight loss occurred through dieting. It seems interesting that the equilibrium associated with point B presents oscillations until convergence is acquired. In that case dieters should, after finishing their diet, gain initially more weight than the weight they lost and later, should converge to their initial weight. Similarly, obese (point C) or anorexic (point A) individuals would require large shocks in their body-mass to converge to an equilibrium such B. Small deviations would bring them back to the saddle-path corresponding to their initial equilibrium and therefore, they would converge back to their initial body-mass. This pattern would explain the necessity of acute treatment to individuals with eating disorders.

Point A is characterized by negative rate of return on body-mass, e.g. $\tilde{H}_{WW}^a < 0$. This means that, even though the individual values food ingestion, she perceives that increasing marginally body-mass has a large negative impact on their current marginal utility throughout the effect of $u_{WW}$. Hence, the individual is willing to eat less and exercise more to obtain larger current utility. Individuals at B are characterized by $\tilde{H}_{WW}^b > 0$. This result means that they choose the equilibrium by providing more relevance to the impact on the modified discount factor, and thus to the future flow of utility, rather than to current flow.
of utility. In fact, those individuals obtain larger utility flow from food ingestion compared to individuals in A, and thus accept larger body-mass. However, they stop accumulating body-mass as they perceive the negative effect on the survival rate, which is much more important to them than the negative impact on their current utility. Finally, individuals in point C are individuals that obtains larger utility flows from food ingestion, compared with individuals at A and B, and thus accumulate intensively body-mass. As individuals in A, they decide to stop accumulating body-mass when they perceive a large negative impact on their current utility due to body-mass (those individuals, given the property of the utility function, face a really large negative marginal utility of weight).

It is clear that individuals in state of anorexia (point A) or obesity (point C) are individuals who behave as “myopic” agents. In fact, the way they choose body-mass is by weighting more intensively their current utility compared to their future utility. Conversely, individuals at B do not have the “myopic” behavior and consider the impact of larger body-mass on their future utility.

Finally, note that in figure 2, even when there are three possible equilibria, only one is chosen. How is the equilibrium chosen? The equilibrium will depend on $W_0$, the initial body-mass. In fact, when $W_0$ is on the neighbor of point A, the long run equilibrium is A and the transition occurs through its saddle-path. Similarly to C and B, while in B the transition occurs through oscillations. Hence in this study the influence of genetics, psychological and social factors can be rationalized by their influence on $W_0$. Once the initial body-mass is set, due to the influence of those factors, individuals’ rational behavior -highlighted in the model above developed- is reflected in the equilibrium body-mass which, in some cases, might be associated with eating disorders.
References


