Twisted sectors in three-dimensional gravity

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Twisted sectors—solutions to the equations of motion with non-trivial monodromies—of three dimensional Euclidean gravity are studied. We argue that upon quantization this new sector of the theory provides the necessary (and no more) degrees of freedom to account for the Bekenstein-Hawking entropy of three-dimensional black holes. [S0556-2821(99)07518-9]

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1. INTRODUCTION

The asymptotic form of three-dimensional anti-de Sitter space,

$$ds^2 \sim r^2 dw d\bar{w} + \frac{dr^2}{r^2},$$

enjoys remarkable properties. (Here $w$ is a complex coordinate related to the spacetime real coordinates $t$ and $\varphi$ by

$$w = \varphi + it.$$  

Note that we work in the Euclidean sector.) The most general set of diffeomorphisms leaving Eq. (1) invariant form a conformal group in two dimensions. This symmetry was discovered by Brown and Henneaux [1] more than 10 years ago; however, only recently it was realized [2] (see also [3]) that it plays a central role in the statistical mechanical description of three dimensional [4,5] black holes.

The metric (1) is a solution to Einstein equations in three dimensions with a negative cosmological constant $\Lambda = -2\ell^2$. The action for Euclidean three-dimensional gravity can be written in the convenient form

$$I_\epsilon[g] = \frac{c}{24\pi} \int \sqrt{g} (R + 2) d\bar{x}^3,$$  

where the constant $c$ is the Brown-Henneaux central charge [1],

$$c = \frac{3\ell}{2 G},$$

and the coordinates $\bar{x}^\mu = x^\mu / \ell$ are dimensionless. In this paper we shall refer all constants to $c$.

The result discovered in [1] states that the asymptotic group of symmetries of Eq. (1) is the conformal group generated by two copies of the Virasoro algebra,\(^1\)

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \]

where the central charge $c$ is the parameter appearing in the action (3).

The first attempt to relate the conformal properties of asymptotic anti-de Sitter space with quantum black holes was done in [7]. It was pointed out in that reference that the degeneracy of states—treating $L_0$ as a number operator—was proportional to the black hole area but with a different coefficient. Strominger [2] made the striking observation that if one regards the conformal algebra as being generated by a $1+1$ unitary conformal field theory, as suggested by the AdS conformal field theory (CFT) correspondence [8–10], then the degeneracy of states for large values of $L_0$ and $\bar{L}_0$ is exactly equal to the Bekenstein-Hawking entropy for the corresponding black hole with mass $M = L_0 + \bar{L}_0 - c/12$ and spin $J = L_0 - \bar{L}_0$. (See [11] and [12] for recent extensions of this idea to higher dimensions.)

In a purely gravitational calculation (like the one suggested in [7]), the boundary CFT is Liouville theory [13] which is a single field and has an effective central charge equal to one. This means that the degeneracy of states grows as \([14,15]\) $\exp(2\pi \sqrt{L_0/6})$ instead of the desired Cardy form $\exp(2\pi \sqrt{CL_0/6})$ which would give the right result. This happens because, contrary to a system of D free bosons with central charge $D$ and a degeneracy growing with $D$, in Liouville theory the central charge is related to the coupling, not the number of fields. Accordingly, the degeneracy does not grow with $c$. In order to have the right counting one would need to have "‘c Liouville fields.’" This has led many authors to conclude that the gravitational degrees of freedom carry only the thermodynamical aspects of the Bekenstein-Hawking entropy [16–18] (see [19] for an earlier discussion of this idea).

In this paper we will show that if one includes solutions to the equations of motions with twisted boundary conditions (on the torus) the degeneracy grows and provides the necessary degrees of freedom to account for the Bekenstein-Hawking entropy. We will show that the degeneracy in the twisted sector is

$$\rho_q(L_0) = \exp(2\pi \sqrt{qL_0/6})$$

\(^1\)This form of the Virasoro algebra is appropriated to the anti-de Sitter sector with $SL(2,C)$ as an exact symmetry. In the black hole calculations we shall encounter a shifted zero mode $L_0 = L_0 - c/24$ appropriated to a torus whose exact symmetries are $U(1) \times U(1)$ [6], and the central term will have the $n^3 \delta_{n+m}$ form.
where $q$, the order of the twisting, is a positive integer. On the other hand, unitarity in the twisted sector will restrict $q$ by $c/q > 1$. Thus, the sector with the maximum degeneracy $q = c$ gives the right Bekenstein-Hawking value. (Note that the sum over $q$ is dominated by its maximum value $q = c$.)

It is important to stress here that the formula (6) has nothing to do with the Cardy formula (which requires modular invariance and unitarity). Equation (6) will follow by counting states explicitly in a given representation of the Virasoro algebra. (In our situation, the sum over representations gives subleading contributions to the degeneracy.)

The results of this paper provide a geometrical justification for the results reported in [20].

II. CLASSICAL SOLUTIONS IN AdS$_3$ GRAVITY

In this section we describe the structure of the classical space of solutions (untwisted and twisted) of three-dimensional anti–de Sitter gravity.

A. AdS$_3$ and Riemann surfaces

The metric (1) with $w = \varphi + it$ is defined on the “cylinder” because $\varphi$ is an angle. It will be useful for our purposes to work on the plane. This is achieved by performing a conformal transformation which also serves as an example to illustrate the conformal properties of Eq. (1). Consider the transformation from the cylinder $w$ to the plane $z$ via

$$z = e^{-iw} \left( 1 + \frac{1}{r^2} + \ldots \right)$$

$$r' = e^{\left( w - \bar{w} \right)/2} r.$$  \hspace{1cm} (7)

It is direct to see that the new metric in the coordinates \{z, r’\} looks, to leading order in $r$, exactly like Eq. (1). This is an example of the invariance of Eq. (1) under a finite conformal transformation. Note that the coordinate $r$ is not invariant under the transformation. In fact, from Eq. (7) we see that $r^2$ transforms like a primary with conformal dimension (1,1). Of course this is necessary if one expects $r^2 dwd\bar{w}$ to be invariant.

It will be convenient to introduce a new radial coordinate $\rho$ defined as $r' = e^\rho$. This change, plus the conformal transformation (7) bring Eq. (1) to the form

$$ds^2 \sim e^{2\rho} dzd\bar{z} + d\rho^2.$$  \hspace{1cm} (8)

This asymptotic metric will be our starting point.

The metrics (1) and (8) are actually exact solutions to the field equations, although their global properties differ. The metric (1) corresponds to the vacuum black hole [4,5], while Eq. (8) to anti–de Sitter space. This can be seen by setting $\lambda = 1/r$ in Eq. (1) and $\lambda = e^-\rho$ in Eq. (8). Both metrics are mapped into the upper Poincare plane. Since $\varphi$ in Eq. (1) is compact there are identifications in this case. The corresponding solution is the vacuum black hole. On the other hand, the coordinate $z$ in Eq. (8) lives in the whole complex plane $z = x + iy$, and the corresponding solution is anti–de Sitter space without any identifications. This change in the topology occurs because in the transition from Eq. (1) to Eq. (8) via Eq. (7) we have kept only the leading terms, ignoring Schwartzian derivative pieces which precisely relate anti–de Sitter space with the vacuum black hole through the shift $L_0 \to L_0 - c/24$.

The three dimensional black hole is asymptotically anti–de Sitter and it approaches Eq. (8) at infinity. In the Euclidean sector, the time coordinate is periodic in order to avoid conical singularities, $t \rightarrow t + \beta$. The topology in the $w$ plane is a torus with the identifications $w \sim w + 2\pi n + 2\pi m \tau (n,m \in \mathbb{Z})$ where $\tau$ is related to the black hole temperature and angular velocity [21,22]. In the $z$-plane this identification reads

$$z \sim e^{-2\pi i \tau} z.$$  \hspace{1cm} (9)

The identification (9) can also be introduced in the anti–de Sitter sector and leads to thermal anti–de Sitter space (TAdS) [23]. In three dimensions, thermal anti–de Sitter space is related to the black hole via a modular transformation [22].

The Euclidean sector of three-dimensional anti–de Sitter space is particularly interesting because the mathematics of Riemann surfaces is available. One can consider generalizations of Eq. (8) on which the coordinate $z$ is defined on a Riemann surface with genus $g$, and the conformal group will still act in a natural way. The solutions to the equations of motion can be summarized in the table

<table>
<thead>
<tr>
<th>2d–solution</th>
<th>3d–solution</th>
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<tbody>
<tr>
<td>sphere ($g=0$)</td>
<td>AdS$_3$</td>
</tr>
<tr>
<td>torus ($g=1$)</td>
<td>black holes/TAdS</td>
</tr>
<tr>
<td>Riemann surface ($g&gt;1$)</td>
<td>?</td>
</tr>
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</table>

(Note that in each case there may be conical or cusp singularities, if sources are added.) We shall see that admitting solutions with non-trivial monodromy properties leads to higher genus solutions, but we shall not discuss this point in any detail. To our knowledge solutions to the Euclidean equations of motion with higher genus (a solid Riemann surface) have not appeared in the literature. (See [24] for a recent discussion about solutions with higher genus in Minkowskian signature.)

B. The untwisted solution

A key property of three-dimensional gravity is the absence of bulk degrees of freedom. This makes it possible to find the general solution to the equations of motion. Under the boundary conditions (8), the general solution has the form of a “travelling wave,”

$$ds^2 = f dz^2 + f\bar{z}d\bar{z} + (e^{2\rho} + f\bar{f} e^{-2\rho}) dzd\bar{z} + d\rho^2,$$  \hspace{1cm} (11)

where $f = f(z)$ and $\bar{f} = \bar{f}(z)$ are arbitrary functions of their arguments [25]. Locally, this metric has constant curvature (as demanded by the vacuum three-dimensional Einstein equations) and, for $\rho \to \infty$, it approaches Eq. (8) with the correct falloff behavior [1].
To prove that Eq. (11) is the most general solution we resort to the analysis of [1]. The most general perturbation of the background metric (8) which preserves its asymptotic structure is constructed by acting on it with a set of asymptotic conformal vectors $\xi^\mu = \xi^\mu (e, \bar{e})$ which depend on two arbitrary functions $e(z)$ and $\bar{e}(\bar{z})$. Acting with these vectors on Eq. (8) one produces the solution (11) with

$$f(z) = \frac{6}{c}T(z),$$

where $c$ is given in Eq. (4). The same formula holds for the anti-holomorphic side. We have named $f(z)$ as in Eq. (12) because the function $T(z)$ transforms under the above symmetry as a Virasoro operator. In the following we shall denote the metric (11) as $ds^2(T, \bar{T})$.

Actually, this construction yields only the pieces linear in $f$ and $\bar{f}$ of Eq. (11) [26]. To obtain the full exact solution (11) we simply note that the term $\int \bar{f}^2 e^{-2\rho} dz d\bar{z}$—which makes Eq. (11) an exact solution—does not affect the leading and first subleading terms. Since three-dimensional gravity does not have any bulk degrees of freedom, it follows that Eq. (11) is the most general solution with the boundary condition (8).

The construction of Eq. (11) by acting on Eq. (8) with the conformal generators parallels the way one constructs representations of the Virasoro algebra. The analogue of the vacuum state $|0 \rangle$ is the anti–de Sitter background (8) which is $SL(2, C)$ invariant. Acting on it with the conformal generators one finds the general solution (11). The metric (11) is then a “descendant” of the metric (8) which is the “primary.” We shall make this point precise below in this section and in Sec. III.

Generically, the perturbations produced on the anti–de Sitter metric (8) via conformal transformations are defined only locally because, apart from the $SL(2, C)$ transformations for which Eq. (8) is invariant, all other conformal mappings are not globally well-defined. In this sense, the metric (8) is a background solution whose local perturbations, consistent with the equations of motion, are described by Eq. (11). In other words, since Eq. (11) has constant curvature, it can be obtained from Eq. (8) via identifications with some discrete group. However, this discrete subgroup will act freely only on some special cases.

The “vacuum” metric with $T = \bar{T} = 0$ is not the only possible background solution. There exists a continuum of metrics for which $T(z)$ has the particular form

$$T(z) = \frac{\Delta}{z^2},$$

where $\Delta$ is some constant. These solutions are also globally well-defined and for suitable values of $\Delta$ they represent black holes. See [25] for the transition from a metric with $T(z)$ of the form (13) to a 3d black hole of mass $M = \Delta + \Delta - c/12$ and spin $J = \Delta - \bar{\Delta}$. We shall denote this particular set of solutions as $ds^2(\Delta, \bar{\Delta})$.

Before leaving this section, we note that the asymptotic conformal symmetries of Eq. (8) can be “lifted” to act on the space of solutions (11). Let $g_{\mu\nu}(T, \bar{T})$ be the metric (11) and $x^\mu = \{z, \bar{z}, \rho\}$. There exist vector fields $\xi^\mu (e, \bar{e})$ depending on two arbitrary (anti-) holomorphic functions $e(z) [\bar{e}(\bar{z})]$ such that under the change $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu (x)$ the metric (11) transforms as

$$g_{\mu\nu}(T, \bar{T}) + \delta g_{\mu\nu} = g_{\mu\nu}(T + \delta T, \bar{T} + \delta \bar{T})$$

with

$$\delta T = \epsilon \partial T + 2 \partial \epsilon T + \frac{c}{12} \partial^3 \epsilon,$$

and $c$ is given in Eq. (4). See [25] for the explicit form of the residual conformal vectors which, of course, coincide asymptotically with the vectors found in [1]. This symmetry is an infinite dimensional symmetry of the space of solutions. In the language of representations of the Virasoro algebra, it maps the members of a conformal family into themselves.

Suppose we start with the metric (8) and perform a finite conformal transformation $z \rightarrow z'(z)$. The finite form of Eq. (15) is known and involves a Schwartzian derivative. Since Eq. (8) can be regarded as a particular case of Eq. (11) for which $T(z) = 0$, making this finite transformation will give a solution of the form (11) with $T'(z') = (c/12) \{z, z'\}$, where $\{, \}$ denotes the Schwartzian derivative. Defining $A(z') = \ln(\partial z/\partial z')$ we find the Liouville type stress tensor,

$$T'(z') = \frac{c}{24}[-(\partial A)^2 + 2 \partial^2 A].$$

The function $A$ can be identified with the holomorphic function appearing in the general solution to the Liouville equation. See [26] for a related approach to obtain the Liouville stress tensor.

Since the residual symmetries of Eq. (11) are nothing but changes of coordinates they can be generated by the constraints of general relativity, supplemented with appropriated boundary terms. In the gauge fixed form of the space of solutions (11), these generators are the functions $T$ and $\bar{T}$ and the associated algebra is the Virasoro algebra [1]. In terms of the modes $L_n$ defined in the usual way,

$$T(z) = \sum_{n=1}^{\infty} \frac{L_n}{z^{n+2}},$$

one obtains the algebra (5).

An important property of these transformations is that $e^{\rho}$ is not a scalar. Indeed $e^{\rho}$ is leading order a primary field with conformal dimension $(1,1)$ and hence $e^{2\rho} dz d\bar{z}$ is invariant. This also explains the presence of the term $e^{-2\rho} T \bar{T}$ in the solution (11). Since $T$ and $\bar{T}$ are quasi-primaries with conformal dimensions $(2,0)$ and $(0,2)$ respectively, the product $T \bar{T}$ has dimension $(2,2)$. The combination $e^{-2\rho} T \bar{T}$ then has dimension $(1,1)$, as needed.
C. The twisted sector

Motivated by the problem of black hole entropy—lack of enough states to account for the large black hole degeneracy, see [15,16,25] for recent discussions—our goal in this section is to generalize the space of solutions described in the last section.

The generalization we have in mind is to admit non-trivial monodromies in the angular direction. In other words, we consider solutions of the form (11) for which the function $f(z)$ has the property,

$$f(e^{2\pi i}z) = g \cdot f(z),$$

(18)

where $g$ is a symmetry of the space of solutions. In Sec. III we shall make this notation precise. For the time being, we consider functions $f(z)$ which have fractional powers in the variable $z$. On the torus (black hole) the angular cycle is non-contractible and thus the twisting does not introduce singularities. On the sphere (AdS) there will be two singularities in the north and south poles of the Riemann sphere.

We shall argue in Sec. IV C that in the Chern-Simons formulation of three-dimensional gravity the twisted solutions arise in a completely natural way.

We consider the solution (11) with a mode expansion for $f(z)$ having fractional powers of $z$. Let $q$ be a non-negative integer, which will be called the order of the twisting. We replace Eq. (12) with

$$f(z) = \frac{6}{c} q T(z)$$

(19)

where

$$T(z) = \frac{1}{q} \sum_{r=0}^{q-1} \sum_{n \in \mathbb{Z}} \frac{L_{m+rl/q}}{zn+rl/q+2}.$$  

(20)

The factors $q$ and $1/q$ appearing in Eqs. (19) and (20) are conventional and included for future convenience. Similar formulas are assumed for the anti-holomorphic factor. The stress tensor $T(z)$ is a finite sum of the form,

$$q T(z) = T_0(z) + T_1(z) + \cdots + T_{q-1}(z),$$

(21)

where $T_r(e^{2\pi i}z) = e^{2\pi i r/q} T_r(z)$. Clearly $T_0(z)$ corresponds to the untwisted solution mentioned in the previous paragraph. The parameter $q$ is regarded as a new degree of freedom in the theory and we shall sum over it.

The modes $L_{m+rl/q}$ can be inverted in terms of $T(z)$ in two steps. First we note that $T_r$ can be expressed in terms of $T$ as,

$$T_r(z) = \sum_{s=0}^{q-1} T_s(z) e^{2\pi isrl/q},$$

(22)

and the modes $L_{m+rl/q}$ are given in terms of $T_r$ as,

$$L_{m+rl/q} = \oint \frac{dz}{2\pi i} e^{zn+rl/q+1} T_r(z).$$

(23)

The metric (11) is still an exact solution because the field equations only see the local properties. For the same reason, the modes $L_{m+rl/q}$ satisfy the same Virasoro algebra (5) but now the label $n$ in Eq. (5) takes values on $Z + rl/q$. One finds the twisted Virasoro algebra,

$$[L_{n+rl/q}, L_{m+sl/q}] = (n - m + (r - s)/q)L_{n+m+(r+s)/q} + c/12 \left((n+rl/q)^2 - 1\right) \times \delta_{n+m+(r+s)/q}.$$  

(24)

This algebra has appeared in [27] in the context of cyclic orbifolds [28].

Note that the modes $L_{n+rl/q}$ are defined such that $0 \leq r \leq q$. The algebra (24) should be understood with the periodicity and reality conditions

$$L_{n+(r+ql)/q} = L_{n+1+rl/q},$$

(25)

$$L_{n-rl/q} = L_{n-1+(q-r)/q},$$

(26)

$$L^*_{m+rl/q} = L_{-n-rl/q}.$$  

(27)

The algebra (24) has a Virasoro sub-algebra (for $r=0$) generating the single-valued conformal transformations of $z$ with central charge $c$. It is clear that this sub-algebra is isomorphic to the Brown-Henneaux generators $L_n$ acting on the single valued, or untwisted, sector.

There is an important point here concerning the value of the central charge appearing in Eq. (24). If one computes the algebra, or operator product expansion (OPE) of the function $T(z)$ defined in Eq. (20) using Eq. (24), one finds a central charge $c/2$. This is correct. The function $T(z)$ contains all sectors $T_r(z)$, $r=0,1,\ldots,q-1$. It is the trivial monodromy generator $T_0(z)$ that needs to be matched with the Brown-Henneaux one, not the full $T(z)$. $T_0(z)$ defines a subalgebra of the full $T(z)$ and satisfies the Virasoro algebra with central charge $c$, as desired. This point will have an important consequence below.

The perturbed metric (11), for a general $T(z)$ with twisted contributions, is no longer periodic under $z\to e^{2\pi i}z$. For the purposes of this paper, this classical non-periodicity will be of no importance because, as we discussed above, the only globally defined metrics are those for which $T(z)$ is of the form (13), for some value of $\Delta$. These metrics are clearly periodic. The constant $\Delta$ (equal to the conformal dimension of the state) is made up of periodic and non-periodic perturbations, but the observable globally defined metric is periodic. We shall come back to this point in the next section.

III. QUANTIZATION

The goal of this section is to study the quantum version of the space of solutions described in the last section.

A. The spectrum

The space of solution described by metrics of the form (11) is infinite dimensional, and so far, completely disorga-
nized. Any function $f(z)$, say $f(z) = z^3$ or $f(z) = z^{1/2}$, provides a solution and there is no way to predetermine their physical properties.

Since there exists a Virasoro symmetry acting on Eq. (11) mapping solutions into solutions, one can classify the solutions using the representations of the Virasoro algebra. In particular, this will enable us to select the physically reasonable ones, e.g., with positive energy.

The idea is to find a map from the vectors in the representation space of the Virasoro algebra, and the space of particular, this will enable us to select the physically reasonable solutions into solutions, one can classify the solutions of the primaries providing a solution and there is no way to predetermine their physical properties.

We identify the function $T(z)$ appearing in Eq. (11) with the expectation value of the quantized operator $\hat{T}$ on a given state $|\Psi\rangle; T(z) = T_{\Psi}(z)$. In other words, instead of considering Eq. (11) with arbitrary values for $T(z)$, we consider only those functions $T_{\Psi}(z)$ which are the expectation values of the quantized $\hat{T}$ on some state $|\Psi\rangle$. This still leaves an infinite dimensional space of solutions, but they are now nicely organized according to the representations of the Virasoro algebra. In particular, if we only consider unitary representations, positivity of the Arnowitt-Deser-Misner (ADM) mass is guaranteed.

Note that since conformal transformations acting on the state $|\Psi\rangle$ produce conformal transformations on $T_{\Psi}(z)$ of the form (15), the identification (31) implements correctly the mapping of solutions into solutions via Eq. (15). Indeed, consider an infinitesimal conformal transformation $z \rightarrow z + \epsilon(z)$. This transformation has an associated group element $U_{\epsilon} = 1 + i\epsilon(z)T(z)$ acting on the representation space: $|\Psi\rangle \rightarrow |\Psi\rangle' = U_{\epsilon}|\Psi\rangle$. It is direct to see that the effect of this transformation on $T_{\Psi}$ is exactly equivalent to Eq. (15).

The map (31) can be extended to the full metric. We extend the above analysis to the anti-holomorphic sector. To each state in the full Fock space $|\Psi\rangle$ there is a corresponding solution to the equations of motion [25],

$$ds^2_{\Psi} = \langle \Psi|d\hat{s}^2|\Psi\rangle,$$

where $d\hat{s}^2$ is the metric (11) on which $T$ and $\bar{T}$ have been replaced by their quantized versions $\hat{T}$ and $\hat{T}$. Note that the metric (11) does not involve products of non-commuting operators$^2$ and thus is well-defined as an operator.

The statistical mechanical picture which follows from the above analysis is the following. We construct the representations of the twisted Virasoro algebra (24). Then we seek (and count) those states which under the map (32) yield a black hole with a given mass and spin. We shall show that the number of these states is enough to account for the Bekenstein-Hawking degeneracy.

Before going to the representations of the algebra (24), let us study the effect of the twisting on the quantum metric. From the algebra (24) it is direct to prove the identity,

$$e^{-\alpha(n+q)}L_n + r/q = e^{\alpha L_n + r/q} e^{-\alpha L_0},$$

where $\alpha$ is a complex number.$^3$ Using this formula with $\alpha$

$^2$This property will not survive in a full gauge invariant quantization. Note, however, that in a Chern-Simons formulation one would need to regulate at most a quadratic function of the currents (Sugawara type).

$^3$The proof is as follows. We consider $X(\alpha) = e^{\alpha L_n + r/q} e^{-\alpha L_0}$. Using the algebra (24) it is direct to see that $X$ satisfies the differential equation $dX/d\alpha = -(n + r/q)X$ whose solution is $X = e^{-\alpha(n+q)}X_0$. The constant $X_0$ does not depend on $\alpha$ and thus can be evaluated on $\alpha = 0$ yielding $X_0 = L_n + r/q$.  

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The associated metric, through Eq. (32), is then univalued and globally defined with \( T(z) \) of the form (13) and \( \Delta = L_0 \).

From Eq. (24), we deduce that \( L_0 \) is a number operator and its value on the state \( |n_i, r_i \rangle \) is given by

\[
L_0 = L_0 - \frac{c}{24} \sum_{i=1}^{s} (n_i - r_i / q).
\]

Note that since we take the vacuum black hole as background metric with \( h = c/24 \), there is a direct relation between the excited levels and \( L_0' \) which is the quantity related directly to the black hole mass and spin, as in Eq. (38), with no added constants.

The sum over \( n_i \) in Eq. (41) is the contribution from the untwisted sector which we know does not give the right degeneracy. The second piece \( r_i / q \) provides a “hyperfine structure” which increases the degeneracy.

Equation (41) can be rewritten in the form,

\[
qL_0' = \sum_i m_i, \text{ with } m_i = qn_i - r_i.
\]

The number of states compatible with a given value of \( L_0 \) is given by the number of combinations of the pair \( n_i, r_i \) such that Eq. (42) is satisfied.

In what follows, the following (trivial) lemma will be of great help. Let \( n \in \mathbb{Z} \) and \( 0 \leq q < c \):

The map: \( \{n, r\} \rightarrow m = qn - r \)

is one to one (and invertible).

Given any \( m \in \mathbb{Z} \) there exists a unique pair \( \{n, r\} \) such that \( m = qn - r \). Then, it follows that all possible combinations of \( \{n_i, r_i\} \) give all possible positive integers \( m_i \) once. The problem of how many states—varying \( n_i \) and \( r_i \)—can exist for a given \( L_0' \) can then be formulated as the number of ways that the integer \( qL_0' \) can be written as a sum of integers \( m_i \). This is of course given by the Ramanujan formula,

\[
\rho \sim e^{\pi \sqrt{qL_0' / 2}},
\]

which is valid for large values of \( qL_0' \). Note the factor of \( q \) multiplying \( L_0' \). Since \( q \) is an integer greater than one, the degeneracy has certainly grown with respect to the pure Liouville calculation which has an affective central charge equal to one. The key step in this counting is that even though \( L_0' \) is not an integer, the combination \( qL_0' \) is.

If we choose \( q = c \), then Eq. (44) gives the expected value for the black hole entropy [2]. Thus, in principle, we have enough states but this is certainly not satisfactory. We would like to prove that \( q \) must not exceed \( c \). We shall now argue that for integer values of the Brown-Henneaux central charge \( c \), unitarity in the representations of the algebra (24) restricts the possible twistings and \( q \) must indeed lie in the range,

\[
1 \leq q \leq c.
\]
Thus, \( q = c \) is the sector with the maximum degeneracy (maximum twisting) and yields the correct Bekenstein-Hawking value. (Note that the sum over \( q \) is dominated by the highest twisting sector \( q = c \).)

This condition arises from a re-ordering of the generators \( L_{n+r|q} \rightarrow Q_m \ (m \in \mathbb{Z}) \) given by

\[
L_{n+r|q} = \frac{1}{q} Q_{qn+r}.
\]

(46)

This is a purely algebraic step. We choose to enumerate the \( L \)'s by integers subscripts according to Eq. (46). A geometrical interpretation for the \( Q \)'s (on the sphere) will be given in the next section.

In view of Eq. (43), it is evident that the map (46) is one to one. The set of all \( Q_n \) with \( n \in \mathbb{Z} \) contains all \( L_{n+r|q} \) and vice versa. Some examples of the above transformation are: \( Q_0 = q L_0, \ Q_1 = q L_{1|q}, \ Q_2 = q L_1, \) etc. The transformation (46) is a particular case of a general procedure studied in [27].

The algebra of the modes \( Q \) can be computed directly from the algebra (24) and yields

\[
[Q_n, Q_m] = (n-m)Q_{n+m} + \frac{c|q}{12} n (n^2 - q^2) \delta_{n+m}.
\]

(47)

This algebra, up to a shift in the zero mode \( Q_0 \) that we discuss below, is a Virasoro algebra with central charge \( c|q \).

It should now be clear where condition (45) comes from. Unitary representations for Eq. (47) exist only for \( c|q > 1 \) thus leading to Eq. (45).

Unitarity is also achieved for a central charge \( c|q \) smaller than one provided it has the Kac form \( c|q = 1 - 6/((m+2) \times (m+3)) \) with \( m = 0, 1, 2, \ldots \). This condition yields a strange quantization condition for \( c \) and, thus for Newton's constant. The degeneracy in this case is more complicated to determine because there are null states in the Verma module. Most likely, the counting will still give the correct degeneracy provided one includes all conformal dimensions producing a modular invariant partition function and using the Cardy formula. Since the central charge is \( c|q \) and \( Q_0 = q L_0 \), one would have \( \sqrt{(c|q)Q_0} = \sqrt{cL_0} \), as desired. We shall not consider this situation here.

The form of the central term in Eq. (47) may seem strange. Note that the transformation (46) maps \( L_{0, \pm 1} \) into \( Q_{0, \pm 1} \) generating an \( SL(2, \mathbb{R}) \) sub-algebra. Let us shift \( Q_0 \) such that the central term in Eq. (47) acquires the usual form with \( Q_0, Q_{\pm 1} \) generating an \( SL(2, \mathbb{R}) \) algebra. This is achieved by redefining the new \( Q_{0}^{\text{new}} \):

\[
Q_0^{\text{new}} = Q_0 - \frac{c}{24} \left( q - \frac{1}{q} \right).
\]

(48)

Let us take the sector \( q = c \). In terms of this new zero mode, unitary representations for Eq. (47) exist provided \( Q_0^{\text{new}} \geq 0 \). From Eqs. (48) and (46) we obtain the bound for \( L_0 \)

\[
L_0 \geq \frac{c}{24} - \frac{1}{24c}.
\]

(49)

For \( c \) large the first term dominates and this is the condition that eliminates (almost all) conical singularities leaving only black holes in the spectrum.

IV. REMARKS

We have exhibited in this article a set of solutions to the vacuum Einstein equations which upon quantization yield the correct Bekenstein-Hawking entropy. In order to have a complete picture of the Hawking evaporation process, we should consider also transitions between states, i.e., emissions of states. This type of process will certainly need other correlation functions beyond the partition function. We shall not consider this problem here. However, we would like to end by making a number of remarks which hopefully will clarify the nature of the solutions that we have presented (at least on the sphere), and may be the starting point towards a dynamical modeling of the evaporation process in 2+1 dimensions.

In particular, in Sec. IV C, we shall argue that the multivalued solutions—twisted sector—arise in a natural way within the Chern-Simons formulation of three-dimensional gravity.

A. The conformal algebra in the covering space

Working with multivalued functions is often problematic and thus it is convenient to pass to the covering space on which the stress tensor is single valued. It is well-known in the theory of complex variables that for any multivalued function of a complex coordinate \( z \), there exists a covering surface for which the given function is single valued. In our case this covering is simply given by the equation

\[
z = u^q.
\]

(50)

This transformation maps the Riemann sphere (AdS3) into itself [up to two singularities in the north and south poles where Eq. (50) has fixed points]. For the torus, this transformation changes the genus [32]. Since we do not have much control on the three-dimensional solutions with higher genus 2D-boundaries, we shall restrict ourselves in this section to the genus zero situation.

We consider the metric (11) with \( f \) given in Eq. (19) and \( T \) in Eq. (20). Let us transform this metric with Eq. (50). Two clarifications are necessary here. First, we shall only transform the complex coordinate \( z \), not the radial coordinate. This means that this transformation is not one of the residual transformations mentioned before and we will obtain a different metric. We could modify Eq. (50) including a radial dependence plus adding the corresponding transformations for \( \rho \) and obtain the same metric defined on the covering space. For our purposes here this is not necessary and would bring unnecessary complications. Second, we shall treat \( T \) classically and therefore we do not include any Schwartzian derivative terms. Of course, if we did include the radial reparametrization, there would be the classical Schwartzian derivative piece.

So, we simply plug Eq. (50) into Eq. (11) and find the new metric.
\[
d s^2 = f du^2 + \ddbar{f} d\bar{u}^2 + \left( q^2 |u|^{2q-2} e^{2\rho} + \frac{f f e^{-2\rho}}{q^2 |u|^{2q-2}} \right) dud\bar{u} + d\rho^2
\]
with
\[
f(u) = \frac{6q}{c} Q(u).
\]

A similar formula is valid for the anti-holomorphic side. The field \( Q \) is simply the conformally transformed version of \( T \) under Eq. (50),
\[
Q(u) = \frac{\partial z}{\partial u}^2 T(u^q)
\]
and is single valued. The modes \( Q_n \) are related to the modes of \( T \) as
\[
L_{n+r/q} = \frac{1}{q} Q_{qn+r}.
\]

These are exactly the generators introduced in Sec. III [see Eq. (46)] which satisfy the Virasoro algebra (47) with central charge \( c/q \). In this context they appear as generators of the conformal symmetry acting on the covering space.

The metric (51) is an exact solution to the equations of motion and it is single-valued. However, it does not satisfy the boundary conditions (8). We shall see in Sec. IV C that working in the Chern-Simons formulation one can easily deal with metrics of the form (51).

How do we recover from Eq. (47), the original Brown-Henneaux algebra? The Virasoro algebra (47) has the symmetry
\[
Q_n \to e^{-2\pi in/q} Q_n.
\]

It is direct to see that the subalgebra of the \( Q \)'s invariant under Eq. (55) is generated by the subset of generators
\[
L_n = \frac{1}{q} Q_{qn}
\]
and they have central charge \( c \). The sub-algebra spanned by the \( L_n \)'s generates the single-valued conformal transformations acting on \( z = u^q \). Thus, they correspond to the Brown-Henneaux generators.

This can also be seen as follows. We start with Eq. (53) and act on this operator with the transformation (55),
\[
Q(u)du^2 = \sum_{n \in Z} \frac{Q_n}{u^{n+2}} du^2 \to \sum \frac{e^{-2\pi i n/q} Q_n}{u^{n+2}} du^2 
\]
Thus, Eq. (55) acts on the plane \( u \) as
\[
u \to e^{2\pi i/q} u,
\]
and the invariant coordinate is, as expected, \( z = u^q \).

**B. The action in the covering space**

We have considered twisted solutions to the equations of motion associated to the action (3), with the boundary condition (8). As shown in the last section, instead of working with a multivalued function \( T \), one can work with a single valued function defined on a different space. In this section we study the action on that space. This sheds some light on the reduction in the central charge in the covering space algebra. Due to conformal invariance, the action in the covering space has the same local form as the action in the original space. However, since the plane \( u \) covers \( q \) times the original plane \( z \), the coupling constant in the covering manifold is \( c/q \) instead of \( c \).

This can be summarized in the following table:

<table>
<thead>
<tr>
<th>Physical spacetime: ( z )</th>
<th>Covering: ( u = z^{1/q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c ) ( \frac{c}{24\pi} \int z^{(R+2)} )</td>
<td>( \frac{c/q}{24\pi} \int u^{(R+2)} )</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>( { L_{n+r/q}, c } )</td>
<td>( { Q_n, \frac{c}{q} } )</td>
</tr>
</tbody>
</table>

It is useful to read this table backwards, from right to left. If we restrict the action in the covering space to the space of functions invariant under Eq. (58) it reduces to an action of single valued functions with coupling \( c \) (recall that the sphere \( u \) covers \( q \) times the sphere \( z \)). This is our starting point and shows that the coupling \( c/q \) in the upper right corner is correct. In the same way, as shown in the last section, the generators \( Q_n \), acting on the invariant subspace reduce to the \( L_n \)'s (\( r = 0 \)).

This analysis posses the question of what is the coupling constant of general relativity. In principle, the action in the plane \( z \) is not preferable in any form with respect to the action in the plane \( u \). We could then argue that \( c/q \) should be related to Newton’s instead of \( c \). The answer to this ambiguity, as in four dimensional gravity, is found in the boundary conditions. We define the action in the plane \( z \) with the boundary conditions (8) as the physical action. The transformation (50) changes the boundary conditions.
C. Chern-Simons formulation and the value of the coupling

The Chern-Simons formulation provides a natural motivation to look for multivalued solutions to the equations of motion. Putting the argument the other way around, within the Chern-Simons formulation, the most natural solution to the equations of motion is Eq. (51), of which Eq. (11) is only a particular case. Thus, when making the conformal transformation (50) that brings Eq. (51) to the asymptotic form (8), one finds the multi-valued solutions introduced above.

We consider here Euclidean three-dimensional gravity formulated as a Chern-Simons theory [33,34] for the group $SL(2,C)$. The level of the Chern-Simons theory will be fixed below.

The Chern-Simons action appropriate to the black hole problem is the covariant one [21,35] supplemented by the chiral boundary conditions,

$$ A^a_z = 0, \quad \bar{A}^a_{\bar{z}} = 0. \quad (60) $$

Recall that $A^a = w^a + ie^a$. At the boundary, the real and imaginary parts of $A^a$ are linked by Eq. (60). The non-zero components $A_z$ and $\bar{A}_{\bar{z}}$ can be written as [25]

$$ A^a_z = 2w^a_z, \quad \bar{A}^a_{\bar{z}} = 2w^a_{\bar{z}}. \quad (61) $$

Thus, Eq. (60) reduces the $SL(2,C)$ gauge field at the boundary to two real currents $A^a_z$ and $\bar{A}^a_{\bar{z}}$ satisfying the affine $SU(2)$ algebra at level $k$.

Conditions (60) are not enough to ensure anti-de Sitter asymptotics [13]. Let $\{A^+, A^-, A^3\}$ and $\{\bar{A}^+, \bar{A}^-, \bar{A}^3\}$ be the components of the $SU(2)_k \times SU(2)_k$ affine currents at the boundary (we omit here the subscripts $z$ and $\bar{z}$). To ensure that the associated metric is asymptotically anti-de Sitter, as in Eq. (8), the boundary conditions (60) must be supplemented with the reduction conditions [13] (see [36] for a different approach)

$$ A^+ = 1, \quad \bar{A}^- = 1, \quad (62) $$

$$ A^3 = 0, \quad \bar{A}^3 \neq 0. \quad (63) $$

This leaves only $L = A^- / k$ and $\bar{L} = \bar{A}^+ / k$ as arbitrary functions. It follows [37] that the affine $SU(2)_k$ algebra reduced by Eqs. (62) and (63) yields for $L$ and $\bar{L}$ the Virasoro algebra with central charge $c = 6k$. Identifying the level $k$ with the usual value $k = 1/4G$ yields $c = 3l/2G$ and confirms that the generators $L = A^- / k$ and $\bar{L} = \bar{A}^+ / k$ are the Brown-Henneaux ones.

As a motivation to a more general situation consider imposing only Eq. (63). The leading piece in the metric with these boundary conditions is easily computed and yields

$$ ds^2 = e^{2\rho} A^+(u) A^-(\bar{u}) dud\bar{u} + d\rho^2. \quad (64) $$

It is clear that further restricting the gauge field with Eq. (62) leads to Eq. (8). However, if we keep them as arbitrary, but fixed, functions we discover that Eq. (64) is precisely the asymptotic behavior of Eq. (51) with

$$ A^+ = qu^{-1}, \quad \bar{A}^- = q\bar{u}^{q-1}. \quad (65) $$

Thus, the Chern-Simons formulation leads to Eq. (51) in natural way.

The next question is what is the algebra leaving Eq. (64) invariant. As a direct generalization of the above case, one can prove that the restriction of the affine $SU(2)_k$ algebra via Eqs. (63) and (65) leads for the combination $Q(u) := (1/k)A^+ A^- = (q/k)u^{-1} A^- (u)$ to the Virasoro algebra with a central charge $c = 6k$. In this case, however, it is not correct to identify $Q$ with the original Brown-Henneaux generators $L$, nor the central term $6k$ with the central charge $4$. The generators $Q$ are the operators introduced in Sec. IV A, acting on the covering space with central charge $c/q$, and leave invariant the generalized boundary conditions (64). We are then led to identify the level $k$ as

$$ 6kq = c = \frac{3l}{2G}. \quad (66) $$

in order to match the center of the algebra (47). Note also that this is in agreement with Eq. (59). The above Chern-Simons theory, with the boundary conditions (64), is appropriate to the action in the covering space with coupling $k = c/6q$.

This issue brings in a problem of interpretation for the calculations of partitions functions that have been done using the Chern-Simons formulation, and its connection to chiral Wess-Zumino-Witten (WZW) models [38,21]. The main assumption on this type of calculation is to quantize the full gauge field $A^a$ without imposing any kind of reduction conditions. However, as we have emphasized above, the values of $A^+$ and $\bar{A}^-$ change the global properties of the spacetime. The coupling $k$ is not only related to Newton’s constant but also to the wrapping of the cover space into spacetime.

There is another curious consequence of Eq. (66). When quantizing a string theory on the AdS$_3$ background, a formula can be given for the spacetime conformal generators
[18]. The associated central charge is $c = 6k_{st}p$ where $k_{st} = l^2/l_s^4$ is the coupling of the $SL(2,\mathbb{R})$ WZW model, $p$ is the winding number of the spacetime coordinate $X(\sigma)$ in the worldsheet, and $l_s$ is the string length. Newton’s constant $G_3$ was associated in [18] to the winding number $p$ via

$$6k_{st}p = \frac{3l}{2G_3}. \quad (67)$$

The similarity between Eqs. (66) and (67) makes it tempting to identifying $k \sim k_{st}$ and $q \sim p$. This amounts to identifying the black hole boundary coordinate $z(u)$ with the string worldsheet coordinate $X(\sigma)$ (compactified on a circle), i.e., treating the covering coordinate as a string worldsheet. The fact that in the quantum gravity approach one quantizes functions of $z$ and thus it corresponds to string field theory has recently been discussed in [39]. Further, we note that in the sector with the maximum degeneracy $c = q$ (see Sec. III), $k$ is of the order of one which is also the value of $k_{st}$ one expects the Correspondence Principle [40] between black holes and string states [41] to work in three dimensions. Finally, the degrees of freedom of three-dimensional gravity, as formulated in [13], are described by a (constrained) non-chiral $SL(2,\mathbb{R})$ WZW model at level $k$. Following this line of reasoning, it would be nice to see an explicit connection between the counting presented in Sec. III, and the string theory approach to black holes.

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