Gravitons and gauge fields in Chern-Simons supergravity

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Despite the nice geometrical properties of higher dimensional Chern-Simons (CS) supergravity theories these actions suffer from one major drawback, namely, their connection with the real world. After some quick remarks on three-dimensional gravity, we consider five-dimensional CS supergravity and study to what extent this theory reproduces the standard low energy description of gravitons and gauge fields. We point out that if one deforms the CS action by changing the value of the cosmological constant by a small amount (thus breaking the CS symmetry), propagation around AdS becomes non-trivial, asymptotically Schwarzschild-AdS solutions exist, and the gauge field acquires its standard quadratic propagator. This is the written version of an invited Lecture delivered at the QG99 meeting, held in Sardinia, Italy, on Sept. 1999.

1. Introduction

Chern-Simons (CS) gravity and supergravity exist in all odd dimensional spacetimes. The three-dimensional case, however, stands apart. There are two reasons for this. First, the 3d CS action is equivalent (up to some issues as the invertibility of the metric) to the standard metric action, as opposed to the higher dimensional CS theory which includes high order powers in the curvature tensor. Second, the 3d theory is special due to its great simplicity. There are many beautiful calculations that can be done yielding interesting and surprising results, most notably, the existence of an asymptotic conformal symmetry [1], and its role in understanding the 3d black hole entropy [2].

Neither the conformal algebra nor its application to black hole physics really need the CS formulation. However, this representation provides simple derivations of some results which are otherwise complicated. As motivation to study the higher dimensional cases, we shall briefly review in Sec. 2 the derivation of the Brown-Henneaux conformal algebra using the CS formulation.

Higher dimensional CS gravitational theories are far less developed than the 3d case. Still, it has been shown that black holes do exist [3], and the asymptotic structure has some of the elements of the 3d case [4] (see, for a recent discussion, [5]). In Sec. 3 we shall study whether 5d CS supergravity has some relation with standard 5d supergravity [28] (which is known to descend from 11d supergravity). We shall point out that if one breaks the CS symmetry by changing the value of the cosmological constant by a small amount, then solutions which are asymptotically AdS-Schwarzschild do exist, and the gauge field entering in the supergravity action acquires its standard propagator.

2. Three-dimensional gravity

Three dimensional gravity has been analysed from many different points of view; we refer the reader to [7–10] for a complete treatment of different aspects of this theory.

In this section we will discuss a short derivation of the Brown-Henneaux conformal algebra [1]. This symmetry plays an important role in 3d gravity and black hole physics [2]. For a recent generalization of the following results to higher dimensions see [11].

Consider the three-dimensional anti-de Sitter line element,

$$\frac{ds^2}{l^2} = e^{2\rho}dw d\bar{w} + dp^2$$

(1)

which will be our background metric. The parameter $l$ is the AdS radius and all coordinates
are dimensionless. In (1), \( w \) and \( \bar{w} \) are coordinates on \( \mathbb{R}^2 \) and \( 0 \leq \rho < \infty \).

The metric (1) can be perturbed preserving the equations of motion as
\[
ds^2 = e^{2\rho} dw d\bar{w} + d\rho^2 + \frac{T(w)}{k} dw^2
\]  
(2)

where
\[
k = \frac{l}{4G}
\]  
(3)
is the 3d (dimensionless) GR coupling constant and \( T(w) \) is an arbitrary holomorphic function of \( w \). It is direct to prove that (2) is still an exact solution to the equations of motion. The most general perturbation preserving the boundary conditions also involves an arbitrary function of \( \bar{w} \) [1] (see [10] for the explicit form of the metric), but for our purposes here this holomorphic version will be quite useful.

In the context of the AdS/CFT conjecture, we expect the perturbations \( T(w) \) to be related to a conformal field theory defined on the complex plane \( w \). To check this, let us act on (2) with a conformal transformation \( w \rightarrow w' \),
\[
w = f(w')
\]  
(4)

where \( f \) is an arbitrary function of \( w \).

The new metric is of course still a solution of the equation of motion, but it changes its form,
\[
ds^2 = e^{2\rho}(\partial' f) dw' d\bar{w}' + d\rho'^2 + \frac{T'(w')}{k}(\partial' f)^2 dw'^2
\]  
(5)
The discovery of Brown and Henneaux [1] is that one can put (5) back into the original form (2) via the following redefinitions of \( \rho \) and \( \bar{w} \),
\[
e^{2\rho} = e^{2\rho'} \frac{\partial f}{\partial f'},
\]  
(6)
\[
\bar{w}' = \bar{w} - \frac{1}{2} e^{-2\rho'} \frac{\partial' f}{\partial f'}.
\]  
(7)

In fact, inserting these new coordinates into (5) one recovers the metric (2) in the primed coordinate system with
\[
T'(w') = T(w)(\partial' f)^2 - \frac{k}{2}\{f, w'\}
\]  
(8)

and where
\[
\{f, w\} = \frac{\partial^3 f}{\partial f^3} - \frac{3}{2} \left( \frac{\partial^2 f}{\partial f^2} \right)^2
\]  
(9)
denotes the Schwarzian derivative of \( f \). The transformation law (8) corresponds to a Virasoro operator with central charge
\[
c = 6k = \frac{3l}{2G}.
\]  
(10)

See [12] for an extensive recent exposition on two dimensional conformal field theory.

Note that when applying the conformal transformation (4) one also needs to change the radial coordinate. This is the Brown-Henneaux version of the IR-UV correspondence discussed in [13].

This derivation of (half of) the Brown-Henneaux symmetry is particularly interesting for two reasons. First, it deals with a finite conformal transformation showing clearly that anti-de Sitter space knows very well the Schwarzian derivative. Second, it acts in a natural way on the space of solutions of the theory mapping solutions into solutions.

Now, there is a very important point that we have omitted so far. We have found the transformation properties of \( T \) via a simple transformation of coordinates. Why should we expect this coordinate redefinition to carry relevant information? The reason is that (4) does not go to the identity at the (conformal) boundary \( \rho \rightarrow \infty \) of anti-de Sitter space. Indeed, \( f \) does not depend on \( \rho \) at all. It is well known that those gauge transformations that do not go fast enough to zero at the boundary may represent non-trivial degrees of freedom.

A powerful way to understand this point is the Regge-Teitelboim approach [14]. The idea is that a proper gauge transformation is generated by a constraint and act on physical states by annihilation \( G|\phi\rangle = 0 \). The state is thus invariant. However, those gauge transformations whose parameters do not vanish fast enough at the boundary are not generated by constraints and it is then inconsistent to assume that they leave physical states invariant. The transformation \( w \rightarrow w' \) is an example of this class of symmetries. A corollary of this result is that two metrics of the form
(2) with different values of $T$ represent physically different solutions to the equations of motion.

This issue, applied to 3d gravity, has been analysed in detail in [1] where it was shown explicitly that the conformal map $w \rightarrow w'$ is not generated by a constraint. We shall now make the transition to the CS formulation and review a simple derivation of this fact. The rest of this section is well-known material and we include it here for completeness. We shall follow mainly [15] and [16], and assume some familiarity with the Chern-Simons formulation of three dimensional gravity [18,19].

We shall first discuss the appearance of an affine Kac-Moody algebra (at the boundary) in any CS theory with a group $G$. The canonical generator $G(\lambda)$ associated to a gauge transformation, $\delta A^a = D\lambda^a = [A^a, G(\lambda)]$, in any three-dimensional Chern-Simons theory is given by [20,15],

$$G(\lambda) = \frac{k}{4\pi} \int \lambda^a F^a - \frac{k}{4\pi} \int_{\partial \Sigma} \lambda^a A^a. \quad (11)$$

Here $\Sigma$ is a two dimensional spatial section with boundary $\partial \Sigma$, $F$ is the 2-form curvature and $A$ the gauge field. The equations of motion of CS theory are $F = 0$ showing that the bulk part of (11) is indeed a constraint, however, the boundary piece does not vanish on shell.

If $\lambda$ is zero and $A$ is finite at $\partial \Sigma$ then the boundary term in (11) is zero. This corresponds to a proper gauge transformation which does not change physical states. But the action is also invariant under transformations with non-zero values of $\lambda$ at the boundary$^1$. The canonical formalism is powerful enough to generate gauge transformation whose parameters do not vanish at the boundary provided the charge

$$J(\lambda) = -\frac{k}{4\pi} \int_{\partial \Sigma} \lambda^a A^a \quad (12)$$

is finite. In this case, the corresponding transformation is not “gauge” because it is not generated by a constraint.$^1$

The charge $J$ is not a gauge invariant quantity because $A$ transforms inhomogeneously under gauge transformations. We act on $J$ with a gauge transformation $\delta A^a = D\rho^a = d\rho^a + [A, \rho]^a$ obtaining,

$$\delta_\rho J(\lambda) = -\frac{k}{4\pi} \int_{\partial \Sigma} ([\lambda, \rho]_a A^a + \lambda_a d\rho^a) = J([\rho, \lambda]) - \frac{k}{4\pi} \int_{\partial \Sigma} \lambda_a d\rho^a \quad (13)$$

On the other hand, as discussed in [14,21,22,1], the variation of the charge under an allowed transformation is itself given by the commutator of two charges,

$$\delta_\rho J(\lambda) = [J(\rho), J(\lambda)]. \quad (14)$$

Equns. (13) and (14) show that the charge $J$ satisfies an affine Kac-Moody algebra with central charge $k/2$. The appearance of an affine algebra at the boundary was first pointed out in [23] in relation to CS theory and the Jones polynomial.

This derivation of the affine algebra is valid for any group $G$. To apply these results to 3d gravity, we consider the particular case $G = \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ and identify the CS coupling $k$ with the gravitational coupling (3).

The next step is to identify the origin of the Brown-Henneaux Virasoro symmetry in this discussion. Given an affine algebra, there is a natural Virasoro operator associated to it via the Sugawara construction. It was shown in [15] that a twisted Sugawara operator yields the correct Brown-Henneaux algebra. The twisting is related to the radial diffeomorphism (6) necessary to bring the metric (5) back to its original form. This derivation is however not complete because it does not fix the value of the central charge. Indeed, from the purely CS point of view, the coefficient in the twisting can be arbitrary and thus the central charge is arbitrary. A full derivation of the conformal algebra in the CS formulation was presented in [16]. The idea is to use the reduction of affine $\text{SL}(2,\mathbb{R})_k$ to Virasoro discussed in [24]. This follows by looking at the metric (2) and choosing boundary conditions for the gauge field such that the associated metric has the asymptotic behaviour (2). These boundary conditions coincide

$^1$Provided some boundary conditions are obeyed. For example, one can impose the chiral boundary condition $A_w = 0$; the residual group contain all parameters which at the boundary only depend on $w$, but are not necessarily zero there.
exactly with the reduction conditions considered in [24]. The Virasoro symmetry then follows in a
direct way with the correct value for the central
charge $c = 6k$.

The details of this derivation can be found in
[6]. See also [25,26] for a comparison with [15].

3. Five dimensional Chern-Simons supergravity

3.1. The action

Contrary to the three-dimensional case, the 5d
Chern-Simons (CS) supergravity action is quite
different from the standard theory. This theory
was first studied in [6]; generalizations to other
dimensions have been studied in [27].

Despite the nice geometrical properties of the
CS construction, this theory has one major draw-
back, namely, its connection with the real world.
For this theory to be “phenomenologically” reason-
able, i.e., related to some compactification of
11d supergravity or $M$ theory, one should be
able to match its field content with the usual
modes of 5d supergravity, namely, the graviton
$h_{\mu\nu}$, a graviphoton $A_\mu$, plus some spinor fields
[28]. While the 5d CS action does contain fields
with those tensorial properties, their propagators
are not the standard ones.

We start by briefly reviewing the construction
of the action [6].

For those familiar with the CS construction and
super Lie algebras, it should be enough to say
that 5d CS supergravity is simply the integral
of a CS density for the supergroup $SU(2,2|N)$. This
supergroup is the smallest super extension
of $SO(4,2)$, which is the group associated to the
pure gravitational part. It is, however, of some
pedagogical value to start from the purely grav-
itational piece and motivate the appearance of
$U(2,2)$ at the bosonic level. We refer the reader
to [6,3,27] for more details. In the following,
we assume some familiarity with the bosonic Chern-
Simons construction.

The purely gravitational piece contains the
$SO(4,2)$ gauge field

$$W_G = e^a P_a + \frac{1}{2} w^{ab} J_{ab}$$

where $P_a$ and $J_{ab}$ are the $SO(4,2)$ generators, and
e$^a$ and $w^{ab}$ are the vielbein and spin connection
respectively. The subscript $G$ refers to “Gravita-
tional”. As usual when studying supergravity, we
consider the particular representation of $SO(4,2)$
obtained via Dirac matrices,

$$P_a = \frac{1}{2} \gamma_a, \quad J_{ab} = \frac{1}{2} \gamma_{ab}, \quad \{ \gamma_a, \gamma_b \} = 2\eta_{ab}. \quad (16)$$

Here $\gamma_{ab}$ is the anti-symmetrised product of Dirac
matrices normalized by $\gamma_{ab} = \gamma_a \gamma_b$ for $a \neq b$. We
use the representation on which,

$$\gamma_0 = i \text{diag}(1,1,-1,-1), \quad (17)$$

and the signature is $\eta_{ab} = \text{diag}(-1,1,1,1,1)$. The
matrix $\gamma_0$ is anti-Hermitian while the others $\gamma_a, \quad a \neq 0$, are Hermitian.

Using the Dirac matrix algebra it is direct to
prove that

$$(\gamma_a \gamma_b) = \gamma_a \gamma_b, \quad (\gamma_a \gamma_b)^\dagger = \gamma_a \gamma_b \gamma_0. \quad (18)$$

Thus, in this representation, the gravita-
tional Chern-Simons field $W_G = (1/2)(e^a \gamma_a + \quad (1/2) w^{ab} \gamma_{ab})$ belongs to the Lie algebra of
$SU(2,2)$ since

$$(W \gamma_0)^\dagger = W \gamma_0, \quad (19)$$

and $\text{Tr} W = 0$. What we have done is to “dis-
cover” that the Lie algebras of $SO(4,2)$ and
$SU(2,2)$ are isomorphic. We note that although
in four dimensions the Chern-Simons action does
not exist, a similar analysis at the level of the Lie
algebras, plus the existence of Majorana spinors,
yields the relation between $SO(3,2)$ and $Sp(4)$.

The group $SU(2,2)$ admits a supersymmetric
extension. The first step is to add an extra
bosonic piece $A$ which induces a non-zero trace
but preserves (19). We consider the extended
$U(2,2)$ bosonic gauge field,

$$W = i A I + \frac{1}{2} e^a \gamma_a + \frac{1}{4} w^{ab} \gamma_{ab} \quad (20)$$

where $A$ is an Abelian gauge field and $I$ is the
identity in the space of Dirac matrices.

A motivation for the field $A$ can be found in [29]
where it was shown that, in the zero cosmologi-
cal constant case, this field is enough to achieve
supersymmetry. If the cosmological constant is
not zero, the structure of the supergroup in five dimensions induces extra bosonic $SU(N)$ gauge fields which however do not couple to the gravitational variables.

The full 5d CS supergravity action has the form [6,27]

$$I_{SU(2,2|N)} = I_{SU(2,2)} + I_{SU(N)} + \text{fermionic pieces + interaction terms.} \quad (21)$$

We shall consider here the situation on which all fermionic and $SU(N)$ gauge fields are zero. The action reduces to,

$$I_{SU(2,2)} = k(I_G + I_A + I_I) \quad (22)$$

where

$$I_G = \int \epsilon_{abcd} \left( \frac{1}{7} R^{ab} \wedge R^{cd} \wedge e^e + \frac{2}{3l^4} R^{ab} \wedge e^e \wedge e^e + \frac{1}{5l^6} e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e \right) \quad (23)$$

is the purely gravitational piece, $I_A = \int AF$ is the Abelian CS action, and $I_I$ the interaction term,

$$I_I = \int \left[ \frac{1}{2} R^{ab} \wedge R_{ab} - d(e_a T^a) \right] \wedge A. \quad (24)$$

From the point of view of CS theory this last term describes the interaction between the gravitational variables and the Abelian gauge field $A$. However, as we shall see soon, the kinetic term for $A$ is actually contained in (24).

$k$ is a dimensionless number representing the CS coupling, and $l$ is the AdS radius. The effective five dimensional Newton constant and cosmological constant are (up to numerical factors),

$$\frac{1}{G} = \frac{k}{l^3}, \quad \Lambda = \frac{1}{l^2}. \quad (25)$$

The gravitational part (23) can be written schematically as,

$$I_G = \frac{1}{G} \int \sqrt{-g} \left( R + \frac{1}{l^2} + l^2 R \cdot R \right) \quad (26)$$

where $R \cdot R$ denotes the Gauss-Bonnet term,

$$R \cdot R = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho}. \quad (27)$$

which in five dimensions is not a total derivative. The first two terms in (26) are of course the usual Einstein-Hilbert and cosmological constant pieces. Our main goal in this note is to analyse whether the quadratic interaction can be discarded or not when expanding around some background.

The motivation to look for phases on which the piece $R \cdot R$ does not contribute is the existence of another 5d supergravity action whose bosonic part has the form [28],

$$I_{Sugra} = \int [\sqrt{-g} (R + F^{\mu\nu} F_{\mu\nu}) + F \cdot F \cdot A]. \quad (28)$$

(The exact coefficients can be found in [28].) This action is known to descend form eleven dimensional supergravity via compactification.$^2$

Besides the obvious differences between (22) and (28), they do have one thing in common, namely, they both contain the graviton $e^\mu_a$ and an Abelian gauge field $A_\mu$ with a CS term. Actually, the name graviton for the field $e^\mu_a$ is somehow premature because, due to the quadratic Gauss-Bonnet interactions, the spin connection is an independent field in this theory [4]. In other words, it is not true that the variation of $I_G$ with respect to $e^\mu_a$ yields an algebraic equation. That equation is in fact differential and carry its own dynamics. (Although $T^a = 0$ is still a solution, at least in vacuum space.)

If we expect (22) and (28) to be related, we will need to resort to some mechanism that can enable us to discard the quadratic interactions in (22). However, even if we could argue that the higher order interactions are negligible in some limit, we will then face the problem that the kinetic piece for $A$ in (22) is not the standard one. In fact, besides the Abelian CS interaction, $A$ does not seem to have a kinetic piece at all! We shall see that these two problems are closely related and that if we can solve the first, the second will be solved automatically.

$^2$It is amusing to note that there also exists a CS action for supergravity in eleven dimensions [27] constructed with the group $OSp(32|1)$ which, by the way, is believed to be the symmetry group of M theory. Somehow, CS theories incorporate in a very natural way the kinematics of M theory but not its dynamics. See [30] for a discussion on 11d CS supergravity and M-Theory.
3.2. Backgrounds

Let us first study whether one could neglect the quadratic interaction terms in (26). We set for the moment $A = 0$ and deal only with (23), or in its metric form (26). As we shall see, this may not be the right way to analyse this problem, but it is a good starting point.

We shall study solutions to the equations of motion with zero torsion. If the quadratic interaction was “small” in some limit, one would expect, for example, to find solutions which asymptotically behave like AdS-Schwarzschild. A situation like this occurs in Born-Infeld electrodynamics whose spherically symmetric solutions approach the standard Coulomb potential. In that case one can argue that at low energies the Born-Infeld action reduces to the standard one. This does not happen in CS gravity.

Solutions with spherical symmetry for the action (22), with $A = 0$, have been studied in [3]. The metric has the form,

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_3$$

with

$$f(r) = -C + \frac{r^2}{l^2}.$$  

The parameter $C$ is a constant of integration, related to the CS-ADM energy of the solution (see below). If $C \neq -1$ there is a curvature singularity at $r = 0$, however, if $C > 0$ this singularity is protected by an event horizon. The metric (30) is a natural generalization to higher dimensions of the three dimensional black hole [17]. We note in particular that the singularity is smoother than the Schwarzschild one. The curvature near $r = 0$ blows up as $1/r^2$ as opposed to Schwarzschild in five dimensions which goes like $1/r^4$. Furthermore, at the equator $\theta_1 = \theta_2 = \pi/2$, the metric (30) is exactly a three-dimensional black hole [17], i.e., 3d anti-de Sitter space with identifications [31].

The Schwarzschild spacetime in five dimensions has the form (29) with $f_{Sch}(r) = 1 + r^2/l^2 - 2M/r^2$. It is clear that $f(r)$ given in (30) does not approach $f_{Sch}(r)$ at large values of $r$. We conclude that asymptotically one cannot neglect the quadratic term. At least not when expanding around the anti-de Sitter background.

The AdS background seems to be the most natural choice for CS theory. However, as stressed in [4,34], it is degenerate because there is no propagation around it. Indeed, the CS equations of motion are,

$$\epsilon_{abcde}(R^{ab} + \frac{1}{l^2}e^a e^b)(R^{cd} + \frac{1}{l^2}e^c e^d) = 0,$$  

and the AdS background satisfies $R^{ab} + \frac{1}{l^2}e^a e^b = 0$. Certainly there can be no linearized perturbations around this background.

It has been shown in [4] that non-degenerate backgrounds for the equations (31) do exists and some examples were displayed explicitly in that reference. However, those examples were rather unphysical and did not have a geometrical motivation.

In this contribution, we will change the point of view. Instead of changing the background, we will change the action. The idea is to deform the equations of motion a little bit such that propagation around AdS becomes non-trivial. In particular, we will modify the value of the cosmological constant. Hopefully, this deformation can be achieved by turning on some of the other fields in the theory and letting them to have a non-zero vacuum expectation value. Work in this direction is in progress [32]. Here we shall motivate this procedure and display its consequences.

We deform the CS equations (31) by adding a little bit of cosmological constant of magnitude $\varepsilon^2$, 

$$\epsilon_{abcde}(R^{ab} + \frac{1}{l^2}e^a e^b)(R^{cd} + \frac{1}{l^2}e^c e^d) + 2\varepsilon = 0(32)$$

where $\varepsilon$ is the deformation parameter. In the deformed theory, there are two “natural” backgrounds related by $\varepsilon \rightarrow -\varepsilon$. We choose $R^{ab} + \frac{1}{l^2}e^a e^b = 0$. Expanding the equations of motion to first order we find

$$2\varepsilon \epsilon_{abcde}\left[\delta R^{ab} + \frac{2(1-\varepsilon)}{l^2}e^a \delta e^b e^c e^d\right] = 0.$$  

This equation coincides exactly with the linearised perturbations following from the standard
Einstein-Hilbert action with a cosmological constant. An important point, however, is the presence of the deformation parameter multiplying (33). This theory has a deformed Newton constant \( G_\varepsilon \) and AdS radius \( l_\varepsilon \) given by

\[
G_\varepsilon = \frac{G}{\varepsilon}, \quad \frac{1}{l_\varepsilon} = \frac{(1-\varepsilon)}{l^2}.
\]

(34)

In particular, we cannot let \( \varepsilon \to 0 \) without losing the contact with standard Einstein equations. We have thus shown that the dynamics of the deformed CS theory expanded around the AdS background is described by the action,

\[
I \sim \frac{1}{G_\varepsilon} \int \sqrt{-g} \left( R + \frac{2}{l_\varepsilon^2} \right). \tag{35}
\]

Let us now consider solutions with spherical symmetry in the deformed theory. Exact solutions for the equations (32) with spherical symmetry and \( \varepsilon \) not necessarily small are known [33]. They have the form (29) with,

\[
f_\varepsilon(r) = 1 + \frac{r^2}{l_\varepsilon^2} - \sqrt{\frac{r^4}{l_\varepsilon^4} \varepsilon^2 + 4MG}.
\]

(36)

If \( \varepsilon = 0 \), we recover the previous solution with \( C = 2\sqrt{MG} \). This solution is indeed asymptotically Schwarzschild since for large values of \( r \), \( f_\varepsilon \) behaves as

\[
f_\varepsilon(r) = 1 + \frac{r^2}{l_\varepsilon^2} - \frac{2MG_\varepsilon}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right).
\]

(37)

where \( G_\varepsilon \) and \( l_\varepsilon \) are given in (34), and \( m = Ml^2 \) is the ADM mass. Thus, asymptotically, the solutions to (32) are indeed solutions to (35).

Before leaving this section we would like to analyse in some more detail the definition of the ADM energy in CS theories and in particular its relation with the integration constant \( M \).

A quick way to analyse this problem is to perform a minisuperspace reduction [3]. Consider the space of metrics of the form (29) for any function \( f \). Let \( f(r) = 1 + h(r) \). Plugging the metric into the equations (32) it is direct to see that the only non-trivial equation is

\[
\mathcal{H} = -\frac{1}{4Gf^{1/2}} \frac{d}{dr} \left[ \left( \frac{r^2}{l^2} - h \right)^2 - \varepsilon^2 r^4 \right] = 0. \tag{38}
\]

This equation is of course trivially integrated and leads to (36). Eq. (38) corresponds to the variation of the CS action with respect to the lapse function \( N = (-g_{00})^{1/2} \). Indeed, \( \mathcal{H} \) is the Hamiltonian constraint which enters in the action as \( H = \int N^2 \mathcal{H} + E \) where \( E \), the ADM energy, is a boundary term added to make \( H \) differentiable [14]. For this class of metrics, \( E \) is given by

\[
E = \frac{1}{4G} \left[ \left( \frac{r^2}{l^2} - h \right)^2 - \varepsilon^2 r^4 \right]_{r \to \infty}. \tag{39}
\]

Here we have adjusted a constant such that \( E = 0 \) on the AdS background \( h_{(AdS)} = (1-\varepsilon)r^2/l^2 \).

The value of \( E \) on the solution (36) is \( E = M \) showing that the parameter \( M \) is indeed the CS-ADM energy.

It is instructive to look separately at each piece in (38) and (39). Consider \( (r^2 - h)^2 = r^4 - 2hr^2 + h^2 \). The piece \( r^4 \) is the contribution to the cosmological constant to the Hamiltonian \(^4\). The piece \( r^2h \) is the contribution from the Einstein Hilbert term, giving a finite contribution for perturbations of the form \( 1/r^3 \). Finally, the piece \( h^2 \) is the contribution to the Hamiltonian coming from the Gauss-Bonnet quadratic interaction.

3.3. The gauge field propagator

We have shown in the last paragraph that the quadratic interactions in the CS Lagrangian, in general, cannot be neglected. However, we have also noticed that a small deformation in the value of the cosmological constant did produce an asymptotically Schwarzschild solution and more generally, a set of perturbations described by standard GR with the “renormalised” couplings (34). This means that asymptotically, the deformed action is controlled by the Hilbert piece and, in that case, one can neglect the quadratic piece, provided one renormalises Newton constant

\(^4\)This piece does not in principle contribute to the energy, however, since we fix \( E_{AdS} = 0 \), the subtraction brings this piece back into \( E \).
appropriately. Deforming the cosmological term means that the CS structure is lost, and in particular, its symmetries are broken. (Actually, only the AdS translational symmetry is broken; Diffeomorphisms and Lorentz rotations are not affected.) We loose the nice geometrical properties but we gain a phenomenological space of solutions, namely, asymptotically Schwarzschild spacetimes.

As further motivation to consider breaking the Chern-Simons symmetry, we shall now show that on this phase, the Abelian gauge field $A$ which enters in the CS action (22) in a rather strange form, becomes the usual $U(1)$ electromagnetic field.

We shall prove that if we discard the terms with more that two derivatives in (24), in particular the Gauss-Bonnet interaction, then the term $d(e_a T^a)A$ appearing in (24) is exactly equivalent to the quadratic propagator $\sqrt{-g}F^{\mu\nu}F_{\mu\nu}$.

The mechanism leading to the appearance of $F^{\mu\nu}F_{\mu\nu}$ through a first order action is actually a standard coupling $\phi F^{\mu\nu}F_{\mu\nu}$ of a scalar field couple to gravity. The metric is not explicitly needed (although present through the tetrad field) and in that sense (40) is topological.

This construction can be generalised to higher dimensional situations involving p-forms. The next case is precisely the action (24) on which we neglect the quadratic pieces in the Riemann curvature. The Chern-Simons interaction $FFA$ does not depend on the spin connection. The variation with respect to the spin connection then yields

$$e_{a\mu} T^a_{\nu\lambda} = \epsilon_{\mu\nu\lambda\rho} \partial^\rho \phi$$

which can be solved for $w^{ab}_\mu$. The spin connection separates in the torsion-free part plus a torsion contribution

$$w_{ab}^\mu = w_{ab}^\mu(e) + e^{\alpha\nu} e^{\lambda\rho} \epsilon_{\mu\nu\lambda\rho} \partial^\phi \phi$$

Here $w(e)$ is the standard formula that follows from $T^a = 0$. We now eliminate $w$ by replacing its on-shell value back in the action. An straightforward calculation shows that one indeed obtains the standard scalar field action coupled to gravity. A quick way to convince ourselves that this is true is by considering the $\phi -$equation of motion following from (40),

$$d(e_a T^a) = 0.$$  \hspace{1cm} (43)$$

Replacing (41) in this equation one derives the correct scalar field equation $\partial_{\mu}(\sqrt{g}g^{\mu\nu} \partial_{\nu}\phi) = 0$.

The key property of (40) is that it does not need the metric tensor explicitly to achieve diffeomorphism invariance. This is a characteristic feature of Chern-Simons Lagrangians. The action (40) can be regarded as a first order formulation of an scalar field couple to gravity. The metric is not explicitly needed (although present through the tetrad field) and in that sense (40) is topological.

The combination $d(e_a T^a) = T^a T_a + R_{ab} e^a \wedge e^b$ is sometimes called the Nieh-Yan invariant. Its relation to chiral anomalies in four dimensions has been discussed in [35].
“torsion” receives this name precisely because it can be absorbed into the spin connection as we have described here.

4. Conclusions and open problems

We have shown in this contribution that deforming five dimensional CS supergravity leads to a new theory which is closer to standard gravitational theories. There are many open problems which we have not touched here. Most importantly, we have not really proved that the deformed theory reduces to the standard supergravity theory, but only checked that the bosonic degrees of freedom have the usual dynamics. It would be very interesting to see whether the full CS supergravity action admits a deformation such that, for some value of the parameter, it reproduces the standard supergravity action. Note that supersymmetry fixes all relative coefficients and thus this would uncover a curious relation between CS and standard supergravity.

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REFERENCES

8. P. Menotti, “Gravity in 2+1 dimensions”, gr-qc/9412017
13. L. Susskind and E. Witten, hep-th/9805114.
32. M. Bañados, work in progress.
34. L.D. Paniak, hep-th/9909112.