Embeddings of the Virasoro Algebra and Black Hole Entropy

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We consider embeddings of the Virasoro algebra into other Virasoro algebras with different central charges. A Virasoro algebra with central charge c (assumed to be a positive integer) and zero mode operator L_0 can be embedded into another Virasoro algebra with central charge one and zero mode operator cL_0. We point out that this provides a new route to investigate the black hole entropy problem in $2 + 1$ dimensions. [S0031-9007(99)08600-7]

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Three-dimensional gravity was put forward by Deser, Jackiw, and 't Hooft [1] as an interesting toy model for gravitational physics. It was then argued in [2] (see also [3]) that it defines a soluble and finite quantum field theory. Questions such as what are the microscopical degrees of freedom giving rise to the Bekenstein-Hawking entropy in three dimensions should then have an answer. However, despite interesting proposals [4,5] it is clear that the answer to this question is not yet available. Even more, the difficulties which arise from trying to provide a consistent quantum description of the black hole entropy have led to the suggestion that Einstein gravity represents a sort of thermodynamical description for the gravitational phenomena and it thus makes no sense to attempt to quantize it (see Ref. [6] and references therein). In the loop representation approach to quantum gravity, successful computations for the black hole entropy, up to a numerical factor, have been achieved (see Refs. [7] and [8]).

Our main tool in analyzing the three-dimensional black hole entropy is the discovery of Brown and Henneaux [9] that the asymptotic symmetry group of three-dimensional anti−de Sitter (adS) space is generated by two copies of the Virasoro algebra with central charge [9]

$$c = \frac{3l}{2G} \tag{1}$$

and, hence, is the two-dimensional conformal group. Here G is Newton's constant; we parametrize the negative cosmological constant as $\Lambda = -1/l^2$ and set $\hbar = 1$. In the semiclassical regime $G \to 0$, c is large. The $2 + 1$ black hole [10] is asymptotically anti−de Sitter and thus the conformal group acts on it in a similar form. However, globally, adS and the black hole differ since the latter is obtained from the former by identification of points. These identifications reduce the exact killing symmetries from $SO(2,2)$ to $SO(2) \times \Re$. For this reason, acting on the black hole, the Virasoro algebra reads [11]

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n^3 \delta_{n+m}\,, \tag{2}$$

having a one dimensional subalgebra generated by L_0. The same holds for the other copy $\bar{L}_n$. We shall call (2) the Ramond Virasoro algebra. The more usual Neveau-Schwarz form of the Virasoro algebra is obtained from (2) by shifting the zero mode as $L_0 = L_0^{\rm NS} - c/24$. The above convention, appropriate to periodic boundary conditions in the spinor field, is natural in the supergravity version of the superconformal algebra [11,12]. The black hole mass and angular momenta are given in terms L_0 and $\bar{L}_0$ as

$$Ml = L_0 + \bar{L}_0, \qquad J = L_0 - \bar{L}_0 \tag{3}$$

with no added constants. With these conventions, anti−de Sitter space has $J = 0$ and $Ml = -c/12$. For $M \geq 0$ the associated metric represents a black hole [10], while the particle solutions studied in [13] have $-c/12 < lM < 0$.

An important open question in $2 + 1$ adS gravity is what is the conformal field theory (CFT) whose energy momentum tensor generates the $c = 3l/2G$ Virasoro algebra. At the classical level (and up to some global issues), this theory is described by a Liouville field [14] (see also [12]) which has two copies of the Virasoro algebra $(L_n, \bar{L}_n)$ with the correct value of c. Since $2 + 1$ gravity has no other degrees of freedom, one would expect that the $2 + 1$ black hole entropy is associated with states in quantum Liouville theory with a given value of $L_0, \bar{L}_0$. The number of such states turns out to be proportional to the black hole area, but with a wrong power of Newton's gravitational constant G. Note that this can happen because in adS_3 gravity there are two length scales, the cosmological radius l and the Planck length parameter $l_p = G$. See Ref. [15] for a first discussion on black hole entropy and the conformal algebra, and see Ref. [16] for a recent review.

A striking observation made in [5] (see also [17]) is that, if the boundary CFT satisfies the following two conditions, (i) $\mathrm{Tr}^{q^{L_0}} \bar{q}^{\bar{L}_0}$ is modular invariant (with $q = \exp(2\pi i \tau)$ and Tr denotes both the character for a given representation and the sum over representations) and (ii) $L_0 \geq -c/24$, $\bar{L}_0 \geq -c/24$ [note that L_0 is the Ramond-Virasoro operator satisfying (2)], then the degeneracy of states for a given value of $2L_0 = lM + J$ and $2\bar{L}_0 = lM - J$ give rise exactly to the Bekenstein-Hawking entropy of a black hole of mass M and angular momenta J. An example of a CFT satisfying these two conditions is a set of c free bosons whose diagonal energy momentum tensor has a central charge equal to c and satisfy both (i) and (ii). However, Liouville theory fails to have the right lower bound on $L_0, \bar{L}_0$ [18] and does not give the right degeneracy [16].

The failure of Liouville theory to provide the right counting has led many authors to conclude that the microscopic origin of the black hole entropy needs an additional structure (probably string theory) not seen by the gravitational field, which would represent only an expectation value for the true quantum fields. Specifically, in the context of the adS-CFT conjecture [19–21], it has been suggested in [6] that the quantum CFT is related to Liouville theory by an expression of the form $\langle T_{\rm CFT}\rangle = T_{\rm Liouville}$. Incidentally, it is worth mentioning here that the adS-CFT conjecture has been used to compute the central charge of $T_{\rm CFT}$ [22,23] and agreement with the Brown-Henneaux value (1) is found.

Further evidence for a string theory description of the black hole was presented in [24], where a string propagating in the adS background was studied. A formula for the spacetime conformal generators was given in terms of the string currents. The spacetime central charge in this approach is associated with the winding of the world sheet current in spacetime. Since the string theory is unitary [for $SR(2,\Re)$ string theories, see Ref. [25] and references therein], the corresponding spacetime CFT is expected to give rise to the right degeneracy. This, to our knowledge, has not been carried out in detail.

Whether string theory is the only solution to this problem or not is not yet clear. However, recent developments in the subject have made it clear that the microscopical description of the three-dimensional black hole entropy requires more degrees of freedom than those arising from a naive analysis of the classical boundary dynamics. [A counter example to this statement is Carlip's [4] original calculation which requires only an affine $SO(2,2)$ algebra, arising in a natural way in $2 + 1$ gravity [26], plus some boundary conditions. The main problems with that proposal seems to be the large number of negative-norm states being counted and the physical meaning of the boundary conditions.]

In this paper we present a new route to attack this problem. We shall add a set of new degrees of freedom to the classical dynamics which, upon quantization, will account correctly for the Bekenstein-Hawking entropy. These new degrees of freedom will have a simple geometrical interpretation although their fundamental quantum origin is not yet known to us. We shall first show our main results and then discuss their significance and possible interpretation.

Let Q_n ($n \in Z$) be a set of operators satisfying the (Ramond) Virasoro algebra with central charge equal to 1,

$$[Q_n, Q_m] = (n - m)Q_{n+m} + \frac{1}{12} n^3 \delta_{n+m}. \qquad (4)$$

Irreducible and unitary representations for this algebra are known and are uniquely classified by a highest weight state $|h\rangle$ with conformal weight $(Q_0 + 1/24)|h\rangle = h|h\rangle$. The shift $1/24$ appears here because Q_0 is the Ramond Virasoro operator entering in (4).

For integer values of the central charge c, the Virasoro algebra (2) is a subalgebra of (4) (see Ref. [27] and references therein for an extensive discussion on this point). Define the generators L_n by

$$L_n = \frac{1}{c} Q_{cn}, \qquad n \in Z, (c \in N). \qquad (5)$$

For $c > 1$ the L_n's are a subset of the Q_n's. Computing the commutator of two L_n's we obtain

$$\begin{aligned} [L_n, L_m] &= \frac{1}{c^2}[Q_{cn}, Q_{cm}] \\ &= \frac{1}{c^2}\left[c(n-m)Q_{c(n+m)} + \frac{c^3 n^3}{12}\delta_{c(n+m)}\right] \\ &= (n-m)L_{n+m} + \frac{c}{12} n^3 \delta_{n+m}, \end{aligned} \qquad (6)$$

as desired. Note that the central charge c of (2) needs to be an integer because otherwise (5) would not make sense. It goes without saying that, if we had started with central charge q in (4), the L_n's defined in (5) would have central charge qc. This raises an ambiguity in the possible embeddings of (2) into (4). In our application to black hole physics we shall favor the $q = 1$ case because it gives the right degeneracy in a siple and natural form. We expect, however, that a deeper understanding of the meaning of the embedding at the level of the gravitational variables will provide a better justification for this choice.

Before going to the black hole application of this result, let us mention some consequences of the above construction [27]. Start with the algebra (4) with central charge $q = 1/2$ (Ising model) and choose $c = 2$. The above construction means that the Ising model has a Virasoro subalgebra with central charge equal to 1. This, of course, can be extended to other situations and may provide unexpected relations between the various conformal field theories [28].

The application of the above result to the black hole problem is direct. We extend the black hole asymptotic algebra (2) by adding new degrees of freedom (new generators) in the way described above such that we pass from (2) to (4). We now look for unitary representations of (4). The number of states (ρ) for a given value of Q_0 grows as

$$\rho(Q_0) \sim \exp(2\pi\sqrt{Q_0/6}). \qquad (7)$$

Note that, since the Q's have central charge equal to one, this formula is correct (for large Q_0). Now, since by construction $Q_0 = cL_0$ [see (5)] the formula (7) does reproduce the right density of states when identifying the values of L_0 and $\bar{L}_0$ with the macroscopic black hole parameters M and J [5].

It is instructive to see how the above mechanism applies in the Euclidean canonical formalism. The Euclidean black hole has the topology of a solid torus [15] whose boundary is a torus with a modular parameter

$$\tau = \frac{\beta}{2\pi}\left(\Omega + \frac{i}{l}\right), \tag{8}$$

where β and Ω are, respectively, the black hole temperature and angular velocity [29,30]. (This definition of τ differs from the one used in [29] in the factor 2π.) The gravitational partition function, under some appropriate boundary conditions and considering only a boundary at infinity, can then be expressed in terms of the character [14,29]

$$Z[\tau] = \mathrm{Tr}\ \exp(2\pi i\tau L_0 - 2\pi i\bar{\tau}\bar{L}_0)\,. \tag{9}$$

where L_0 is the zero mode Virasoro operator in (2).

The expected behavior in the semiclassical Gibbons-Hawking (GH) approximation for Z is (see [31] for a recent discussion)

$$\ln Z_{\mathrm{GH}}(\beta) = \frac{\pi^2 l^2}{2G\beta}\,, \tag{10}$$

where we have set $\Omega = 0$ (no angular momentum) for simplicity. This follows from evaluating the black hole free energy $-\beta F = -\beta M + S$ with $M = r_+^2/(8Gl^2)$, $S = 2\pi r_+/(4G)$, and $\beta = 2\pi l^2/r_+$.

The question of the degeneracy of states can now be reformulated as to whether the partition function (9) reproduces or not this semiclassical behavior.

If $Z[\tau]$ defined in (9) was modular invariant with $L_0, \bar{L}_0 \geq -c/24$, and c given in (1), then one can see directly that $Z[\tau]$ would behave exactly as (10) [32]. This is nothing but the canonical version of the results obtained in [5]. The trouble is that for $c > 1$, either looking at representations of the Liouville theory or the Virasoro algebra (2) itself, it is not possible to fulfil both conditions. Indeed, (9) does not show the behavior (10). The trace in (9) needs to be taken on a bigger Hilbert space; more degrees of freedom are necessary.

Let us take the full algebra generated by the Q's, of which (2) is a subalgebra, and compute the trace over representations of (4) with central charge one. We use (5) and write

$$Z_Q[\hat{\tau}] = \mathrm{Tr}_Q \exp(2\pi i\hat{\tau}Q_0 - 2\pi i\hat{\bar{\tau}}\bar{Q}_0)\,. \tag{11}$$

with $\hat{\tau} = \tau/c$. We have thus replaced the Virasoro character (ch) with central charge c and modular parameter τ, with a different character with central charge one and modular parameter τ/c. In symbols,

$$\mathrm{ch}(c,\tau) \rightarrow \mathrm{ch}(1, \hat{\tau} = \tau/c)\,. \tag{12}$$

The character (11) can be computed exactly. After an appropriate sum over zero modes (see below), we find [33]

$$Z_Q[\hat{\tau}] = \frac{A}{\mathrm{Im}(\hat{\tau})^{1/2}|\eta(\hat{\tau})|^2} \tag{13}$$

which is invariant under modular mappings acting on $\hat{\tau}$. A is a constant which does not depend on τ. The asymptotic behavior of (13) can be determined either by using the well-known asymptotic expansions for the Deddekind function or by looking at (11) and using modular invariance, as done in [32]. In any case, one finds

$$\ln Z_Q \sim \frac{i\pi}{6\hat{\tau}} \tag{14}$$

which, in terms of τ, reproduces exactly the Gibbons-Hawking approximation (10). We then see a complete analogy between the relations $Q_0 = cL_0$ (microcanonical) and $\hat{\tau} = \tau/c$ (canonical) which encode the addition of the new degrees of freedom. In particular, note that the semiclassical approximations Q_0 large and $\hat{\tau}$ small are controlled by the central charge c, without imposing any conditions over L_0, $\bar{L}_0$ or τ. This means that the asymptotic behaviors (7) and (10) are actually universal for all values of M and J since the semiclassical condition $c \gg 1$ ensures both Q_0 large and $\hat{\tau}$ small [34].

It is important to mention here that the exact result (13) for the partition function arose after an integration over zero modes (see [33]). This integration is actually not necessary to have the right semiclassical limit. Indeed, for each representation with conformal weight h, the character approaches (10) for small β. We have chosen to perform the integration over h in order to find a modular invariant partition function, which is likely to be an important property of the black hole dynamics. Making the integral over the conformal dimensions is also a statement on the spectrum of the associated CFT. From the geometrical point of view, we know that positive values of $L_0 + \bar{L}_0$ represent black holes, while $-c/12 < L_0 + \bar{L}_0 < 0$ give rise to conical singularities. The states considered in the computation of (13) have $Q_0 + 1/24 = h + N$, and we have integrated over all positive values of h. This means $Q_0 \geq -1/24$. In terms of L_0 this implies $L_0 \geq -1/(24c)$. Thus, the modular invariant partition function (13) does contain states corresponding to the particles studied in [13]. However, curiously, not all particle masses are allowed but only the small region $-1/(12c) < L_0 + \bar{L}_0 < 0$. In particular, anti–de Sitter space with $L_0 + \bar{L}_0 = -c/12$ is not included.

The issue of modular invariance brings into the scene another important point. The boundary of the black hole is a torus with a modular parameter τ, and one could have expected the partition function to be modular invariant under modular mappings acting on τ. However, we have found that Z is invariant under the modular group acting on $\hat{\tau} = \tau/c$. This is not surprising since all we have done is use the identity $\tau L_0 = \hat{\tau}Q_0$ and compute the trace over representations of Q_n. [This is summarized in (12).] If correct, this scaling of the modular parameter should have a precise meaning to be uncovered. In particular, one would like to know whether the addition of the new generators, giving rise to (4) and $\hat{\tau}$, could be understood in terms of the gravitational variables themselves, up to some rescaling or duality transformations, or if a more sophisticated mechanism such as introducing string degrees of freedom is necessary.

In summary, we have shown that a Virasoro algebra with central charge c (c: integer) can be understood as a subalgebra of another Virasoro algebra with central charge one. We have thus imposed the quantization condition $c \in N$, where c is the central charge (1), and have extended the Brown-Henneaux conformal algebra by adding new generators. The new conformal algebra reproduces in a natural way the semiclassical aspects of the $2 + 1$ black hole thermodynamics. An important open question not addressed here is the uniqueness of this approach. The embeddings of the Virasoro algebra studied here are of course not unique, although it is not clear to us that other embeddings of (2) will give rise to the right entropy. At any rate, a more detailed calculation of other semiclassical quantities such as decay rates [35] in the Q theory should test its uniqueness and correctness.

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[31] M. Bañados and F. Méndez, Phys. Rev. D **58**, 104 014 (1998).
[32] The proof is as follows: Let $\Omega = 0$, then from (8) $\tau = i\beta/(2\pi l)$. We first use the symmetry of Z under $\tau \to -1/\tau$ and write $Z[\beta] = \mathrm{Tr}\exp[-4\pi^2 l(L_0 + \bar{L}_0)/\beta]$. In the limit $\beta \to 0$ (large black holes) this sum is dominated by the lowest eigenvalue of $L_0 + \bar{L}_0$ which is $-c/12$. By replacing this value in $Z[\beta]$ and using (1), one obtains (10).
[33] The key steps here are the following: Take a highest weight state $|h\rangle$ with $(Q_0 + 1/24)|h\rangle = h|h\rangle$ and construct the associated descendants states $|n_1, ..., n_r\rangle = Q_{-n_1} \cdots Q_{-n_r}|h\rangle$. For a given h, one has (see, for example, Ref. [36], p. 158) Tr $q^{Q_0}\bar{q}^{\bar{Q}_0} = q^h \bar{q}^{\bar{h}}/|\eta(\tau)|^2$ with $q = \exp(2\pi i \hat{\tau})$, and $\eta(\tau)$ is the Deddekind function. Now take $h = \bar{h} = x^2$ and integrate overall values of x; the result (13) is reproduced. Note that the sum over x is quite natural when working in the Fock space representation.
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