Dilaton gravity (with a Gauss-Bonnet term) derived from five-dimensional Chern-Simons gravity

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We study the problem of boundary terms and boundary conditions for Chern-Simons gravity in five dimensions. We show that under reasonable boundary conditions one finds an effective field theory at the four-dimensional boundary described by dilaton gravity with a Gauss-Bonnet term. The coupling of matter is also discussed. [S0556-2821(97)00204-X]

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The existing link between three-dimensional (3D) Chern-Simons theory and two-dimensional conformal field theory [1] is a remarkably powerful tool in the applications of Chern-Simons (CS) theory to 2+1 gravity. Carlip [2] has shown that the number of states of a conformal field theory lying at the horizon gives the correct Bekenstein-Hawking entropy. Also, the rich asymptotic structure of anti-de Sitter 2+1 gravity [3] can be analyzed in a simple way as a Wess-Zumino-Witten (WZW) model lying at the boundary [4–6].

Chern-Simons theories exist in all odd-dimensional spacetimes. It is therefore a natural question to ask whether they induce a "conformal" field theory in a lower dimension. Steps in that direction were taken in [7] where a generalization to four dimensions of the WZW action and its associated current algebra was constructed. The WZW 4 theory was also shown to be related to a Kahler-Chern-Simons theory. Recently [8], the precise connection between pure Chern-Simons theory and the four-dimensional current algebra found in [7] was established by studying the problem of boundary terms and boundary conditions in Chern-Simons theory for groups of the form $G \times U(1)$.

We consider in this paper the problem of boundary conditions and boundary terms in a particular five-dimensional Chern-Simons theory which is the analogue of the Chern-Simons formulation of 2+1 gravity studied in [9,10]. We shall see that by imposing natural boundary conditions on the five-dimensional problem one obtains a four-dimensional field theory described by dilaton gravity with a Gauss-Bonnet term.

(i) Five-dimensional Poincaré Chern-Simons gravity. We consider a five-dimensional Chern-Simons theory for the group ISO(3,2) [or ISO(4,1)] defined by the action

$$I_{CS} = \frac{1}{2} \int_M \epsilon_{ABCDE} \tilde{R}^{AB} \wedge \tilde{R}^{CD} \wedge e^E,$$  \hspace{1cm} (1)

where $\tilde{R}^{AB} = dW^{AB} + W^A_C \wedge W^{CB}$ is the five-dimensional curvature two-form. The fields $W^{AB}$ and $e^A$ can be collected together to form an ISO(3,2) [ISO(4,1)] connection, and Eq. (1) can be shown to be an ISO(3,2) [ISO(4,1)] Chern-Simons action [11–13]. Indeed Eq. (1) is explicitly invariant under SO(3,2) [SO(4,1)] rotations. It is also invariant, up to a boundary term, under the Abelian translation $\delta e^A = \nabla A^A$. Moreover, factor out a scale factor $\delta W^{AB} = 0$. Hence, Eq. (1) is invariant under ISO(3,2) [ISO(4,1)] provided one imposes some appropriate boundary conditions. For this reason, we call Eq. (1) the Poincaré Chern-Simons action. The action (1) is a natural extension of the 2+1 Chern-Simons gravity action studied in [9,10] for zero cosmological constant.

Our notations are the following: $\gamma_{AB} = \text{diag}(-1,1,1,1)$; capital indices $A,B,…$ run over $\text{SO}(3,2)$ if $\sigma = -1$ and over $\text{SO}(4,1)$ if $\sigma = 1$. $M$ is a five-dimensional manifold with the topology $\Sigma \times \mathbb{R}$ and $\Sigma$ has a boundary denoted by $\partial \Sigma$ (see Fig. 1). We shall call $B$ the "cylinder" formed by the direct product of $\mathbb{R}$ and $\partial \Sigma$.

$$B = \partial \Sigma \times \mathbb{R}.$$  \hspace{1cm} (2)

According to Fig. 1, the surface $B$ is represented as the hypersurface $x^5 = \text{const}$. Local coordinates on $B$ are denoted by $x^M$, $M = [0,1,2,3,4]$; local coordinates on $\Sigma$ are denoted by $x^\mu$, $\mu = [0,1,2,3]$; and local coordinates of $\Sigma$, for all times, are denoted by $x^i$, $i = [1,2,3,4]$. The SO(3,2) [SO(4,1)] covariant derivative is denoted by $\nabla$. Varying Eq. (1) with respect to all the fields (assuming that the boundary terms cancel out) we obtain the five-dimensional equations of motion

$$\epsilon_{ABCDE} \tilde{R}^{AB} \wedge \tilde{R}^{CD} \wedge e^E = 0,$$  \hspace{1cm} (3)

$$\epsilon_{ABCDE} \tilde{R}^{AB} \wedge \tilde{T}^C = 0,$$  \hspace{1cm} (4)

where $\tilde{T}^A = \nabla e^A$ is the five-dimensional torsion two-form. The configuration space defined by the above equations is stratified into different regions with a different number of
degrees of freedom. For example, the configuration \( \tilde{R}^{AB} = 0 = T^A \) solves the above equations but carries no local degrees of freedom. There exist, however, other solutions for which the curvatures do not vanish and represent propagating modes. The maximum number of (physical) local degrees of freedom for this theory is shown to be 13. We refer the reader to [8] for details on this point. It is enough for our purposes here to mention that on the ray of phase space that carries the maximum number of degrees of freedom, the above equations can be written in the useful form [8]

\[
\frac{de^A_i}{dt} = D_i e^A_0 + N^A T_{ki}, \tag{5}
\]

\[
\frac{dW_i^{AB}}{dt} = D_i W_0^{AB} + N^A \tilde{R}^{AB}_{ki}, \tag{6}
\]

\[
0 = \epsilon^{ijkl} e_{ABCDE} \tilde{R}^{AB}_{ij} \tilde{R}^{CD}_{kl}, \tag{7}
\]

\[
0 = \epsilon^{ijkl} e_{ABCDE} \tilde{R}^{AB}_{ij} \tilde{R}^{CD}_{kl}, \tag{8}
\]

where we have split \( W_0^{AB} = W_0^{AB} dt + W_i^{AB} dx^i \) and \( e^A = e^A_0 dt + e^A_i dx^i \). Note that Eqs. (5) and (6) explicitly show that the time evolution is generated by a gauge transformation with parameter \( \{ W_0^{AB}, e^A_0 \} \) plus a spatial (improved [14]) diffeomorphism with parameter \( N^A \). As was shown in [8] the normal deformations, or timelike diffeomorphisms, do not represent an independent symmetry. This is similar to what happens in 2+1 dimensions where the full diffeomorphism invariance is contained in the gauge group [10]. Note that the vector \( N^A \) did not appear in the original Lagrangian. It appears here because in obtaining Eqs. (5) and (6) from Eqs. (3) and (4) a degenerate matrix was inverted. The vector \( N^A \) parametrizes the linear combination of null eigenvectors of that matrix [8].

To integrate Eqs. (5) and (6) we only need to give Cauchy data on the initial hypersurface. The initial data, on the other hand, must satisfy the constraints Eqs. (7) and (8). It is easy to see from Eqs. (5) and (6) that the time evolution preserves the constraints; that is, a configuration satisfying Eqs. (7) and (8) at \( t = t_1 \) will remain on that surface at late times. Equations (5) and (6) can be studied, for example, in the gauge \( e^A_0 = 0, W_0^{AB} = 0, \) and \( N^A = 0 \). In that gauge we find that \( e^A_i \) and \( W_i^{AB} \) are time independent; thus, given their initial values, they are known for all times.

(ii) Boundary terms. The above equations of motion define an extremum for the action principle provided some suitable boundary conditions are imposed such that all boundary terms cancel out. The boundary term coming from the variation of the action (1) is easily computed obtaining

\[
- \int_{\partial M} e_{ABCDE} \tilde{R}^{AB} \wedge e^C \wedge \delta W^{DE}. \tag{9}
\]

The boundary \( \partial M \) has three connected components (see Fig. 1),

\[
\partial M = \Sigma_1 \cup \Sigma_2 \cup B. \tag{10}
\]

On the initial and final boundaries (denoted by \( \Sigma_1 \) and \( \Sigma_2 \) in Fig. 1), a natural way to cancel Eq. (9) is by imposing \( \delta W_i^{AB} = 0 \). Of course, the possible values, or Cauchy data, for \( W_i^{AB} \) at \( t = t_1 \) and \( t = t_2 \) must satisfy the constraints (7) and (8). Also, as a result of the first-order character of the equation for \( W_i^{AB} \) its value on the final surface is—classically, but not quantum mechanically—related to the data given on the initial surface. This is similar to what happens in the example of a free particle in the momentum representation. Since the momentum is conserved, classically one cannot prescribe two different values for \( p_1(t) \) at \( t = t_1 \) and \( t = t_2 \); otherwise, there will be no solutions. Quantum mechanically, however, \( p_1 = p(t_1) \) and \( p_2 = p(t_2) \) are completely independent but the propagation amplitude for going from \( p_1 \) to \( p_2 \) is shown to be proportional to \( \delta(p_1 - p_2) \).

There is another boundary term appearing at a fixed value of \( x^5 \) for all \( t \) (denoted by \( B \) in Fig. 1). On that surface, \( W_i^{AB} \) cannot be fixed to an arbitrary value because that would overdetermine the variational principle. Moreover, the action must define a propagation amplitude (through the path integral) from an initial to a final surface, without needing to prescribe data on the intermediate states.

There are many different ways to proceed in this situation. Perhaps the most natural way to treat this boundary term is by imposing appropriated boundary conditions such that Eq. (9) can be written in the form \( \delta X \); then, one subtracts \( X \) from the initial action, thus producing a new action whose variation does not generate any boundary terms [15]. We shall carry out this procedure below.

Another method to deal with boundary terms would be to pass to the Hamiltonian formalism. The Hamiltonian methods developed in [8] appear to be particularly appropriate to handle this case because a \( U(1) \) field, necessary to apply the results of [8], can be coupled in a natural way to the action (1) [13]. Thus, one expects to find a four-dimensional current algebra at \( B \), with a Kahler form equal to the \( U(1) \) field strength [8]. The presence of a current algebra at \( B \) opens the possibility to find a statistical-mechanical description [2] for the entropy of the five-dimensional black holes found in [12]. We shall not attack this problem in this paper.

Finally, a third possibility to cancel Eq. (9) is simply to impose the coefficient of \( \delta W_i^{AB} \) in Eq. (9) to be zero. Note that this is similar to the open string (Newmann) boundary conditions where \( \partial_x x^\mu \) is set equal to zero in order to cancel the boundary terms at the spatial boundaries. In this paper we shall mainly consider this latter possibility and study the resulting theory at \( B \).

We thus impose the field equation at \( B \),

\[
e_{ABCDE} \tilde{R}^{AB} \wedge e^C = 0. \tag{11}
\]

Note that, in order to simplify the notation, we have used the same symbols to represent the five-dimensional forms and their four-dimensional projections. The above equation is a four-dimensional equation with \( \tilde{R}^{AB} = \frac{1}{2} \tilde{R}^{AB}_{\mu \nu} dx^\mu \wedge dx^\nu \) and \( e^A = e^A_\mu dx^\mu \) where \( x^\mu (\mu = [0,1,2,3]) \) are local coordinates on \( B \). In terms of local coordinates Eq. (11) reads

\[
e_{\mu \nu \lambda \rho} e_{ABCDE} \tilde{R}^{AB}_{\mu \nu} e^C = 0. \tag{12}
\]

Before going into the analysis of the consequences of Eq. (12) it is necessary to check whether this equation completely defines the dynamics of the fields at \( B \). In other
words, we would like to know if the projections of Eqs. (3) and (4) to B become identities after Eq. (12) is imposed. It turns out that Eq. (4) (projected to B) is indeed satisfied as a consequence of Eq. (12), but the same is not true for Eq. (3). The projections of Eqs. (3) and (4) to B are, respectively,

\[ e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} = 0, \]

\[ e^{\mu\lambda\nu\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\lambda\nu\rho} = 0. \]

Taking the \( \nabla_\rho \) covariant derivative (tangential to B) of Eq. (12) and using the Bianchi identity \( e^{\mu\nu\lambda\rho} \nabla_\nu \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\lambda\rho} = 0 \), one obtains Eq. (14). Thus, Eq. (14) is indeed identically satisfied once the boundary condition (12) is imposed.

The situation is not so simple with Eq. (13). Equation (13) is quadratic in the curvature while Eq. (12) is linear. To see whether Eq. (13) is a consequence of Eq. (12) we multiply Eq. (12) by \( R_{\mu\nu}^{\alpha\beta} \). Using some simple combinatorial identities one obtains the equation

\[ e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} = 0. \] (15)

This equation is almost what we need. Indeed, the coefficient of \( e^{5}_\sigma \) in Eq. (15) is exactly Eq. (13), but one cannot infer from Eq. (15) that Eq. (13) holds because \( e^{5}_\mu \) is not a squared matrix and thus it cannot be inverted. However, the five-dimensional vielbein \( e^{A}_\mu \) is invertible and therefore \( e^{A}_\mu \) has rank 4. This means that Eq. (15) does imply the vanishing of four of the five equations (13). To isolate the part of Eq. (13) not contained in Eq. (15) we need to break the explicit five-dimensional covariance.

We decompose the five-dimensional indices \( A, B, \ldots \) into four-dimensional indices \( a, b, \ldots \) \([a = (0, 1, 2, 3)]\) and write \( e^{A}_\mu \) and \( W^{AB}_\mu \) in the form

\[ e^{A}_\mu = (e^{a}_\mu, e^{5}_\mu), \quad W^{AB}_\mu = (w^{ab}_\mu, w^{a5}_\mu). \] (16)

Note that under this decomposition all the fields transform in a definite way under the Lorentz group \( SO(3, 1) \). Indeed, both \( e^{a} \) and \( W^{a5}_\mu \) transform as vectors under \( SO(3, 1) \); \( e^{5} \) is a Lorentz scalar and \( w^{ab}_\mu \) is a Lorentz connection. We can thus already identify the elements of general relativity, namely, the tetrad \( e^{a}_\mu \) and the spin connection \( w^{ab}_\mu \). In this notation, Eq. (15) takes the form

\[ e_{abcd} e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} e^{5}_\sigma - 2 \tilde{R}_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho} e^{5}_\sigma = 0. \] (17)

If we borrow from Eq. (13) the single equation \( e_{abcd} e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} = 0 \), then invertibility of \( e^{5}_\mu \) together with Eq. (17) implies \( e_{abcd} e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} = 0 \). We have thus isolated the part of Eq. (13) not contained in Eq. (11), namely, the equation

\[ e_{abcd} e^{\mu\nu\lambda\rho} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\alpha\beta\gamma\delta}^{\mu\nu\lambda\rho} = 0. \] (18)

This equation contributes in a nontrivial way to the dynamics at \( B \).

The independent field equations at \( B \) are thus Eq. (11) plus Eq. (18). Now, one would like to know whether these equations define a sensible field theory at \( B \). One faces an immediate problem because the number of equations and number of fields at \( B \) do not match. Indeed, the number of fields is 60 \([\#(W^{AB}_\mu) = 40 \text{ plus } \#(e^{5}_\mu) = 20]\) while the number of equations is 41 \([\text{there are } 40 \text{ equations in Eq. (11) plus Eq. (18)}]\). One may wonder whether the remaining 19 fields may be a signal of some extra gauge symmetry. This is not the case because we have counted here all the equations including constraints. If there was some extra gauge symmetries we should also find their associated constraints. (The number of dynamical fields plus Lagrange multipliers must equal the number of dynamical equations plus constraints.) A second possible interpretation for those fields is as matter fields. This is certainly an interesting possibility. Indeed, in Kaluza-Klein theories this is exactly the mechanism by which matter is brought in. In this particular case, however, we have found no natural interpretation in that sense.

To remedy the mismatch of fields and equations we propose to impose further conditions on the fields at \( B \). We shall not attempt to give a general set of extra possible boundary conditions. Rather, we shall exhibit a set of conditions which leave the right number of independent fields and the resulting theory is dilaton gravity with a Gauss-Bonnet interaction.

We impose at \( B \) the 19 restrictions

\[ e^{5}_\mu = l \partial_\mu \varphi \quad \text{and} \quad W^{a5}_\mu = \frac{e^{a}_\mu}{l}, \] (19)

where \( \varphi \) is a dimensionless Lorentz scalar and \( l \) is an arbitrary constant with dimensions of length. The independent fields are then reduced to the four-dimensional tetrad \( e^{a}_\mu \), the spin connection \( w^{ab}_\mu \), and the dilaton, \( \varphi \). The number of independent fields is thus 41, as required. Note that since \( e^{5} \) is a Lorentz scalar and both \( e^{a} \) and \( W^{a5}_\mu \) are Lorentz vectors, conditions (19) are Lorentz invariant. The local symmetry group of the boundary conditions has thus been broken down to the Lorentz group.

Some useful identities that hold under Eqs. (19) are

\[ \tilde{R}^{ab}_\mu = R^{ab}_\mu - \frac{\sigma}{l} e^{a}_\mu \wedge e^{b}_\mu, \] (20)

\[ \tilde{R}^{5}_\mu = T^{a}_\mu, \] (21)

\[ \tilde{T}^{a}_\mu = T^{a}_\mu + e^{a}_\mu \wedge d \varphi, \] (22)

\[ \tilde{T}^{5} = 0, \] (23)

where \( D \) is the \( SO(3, 1) \) covariant derivative, \( R^{ab}_\mu \) the \( SO(3, 1) \) curvature two-form and \( T^{a}_\mu = D e^{a}_\mu \) the four-dimensional torsion. We also recall that \( \sigma \) is the signature of the fifth dimension of the local group: \( \eta_{AB} = \text{diag}(−1, 1, 1, 1, \sigma) \). Hereafter we shall use the notation of differential forms. It is understood that they are forms defined at \( B \).

Replacing Eqs. (19) into (11) and using the above identities one obtains the \( SO(3, 1) \) equations

\[ e_{abcd} \tilde{R}^{ab}_\mu \wedge e^{c}_\mu = 0, \] (24)

\[ e_{abcd} (2 \sigma T^{a}_\mu \wedge e^{b}_\mu - l^2 \tilde{R}^{ab}_\mu \wedge d \varphi) = 0, \] (25)

plus Eq. (18) which reads
\[
\epsilon_{abcd} \tilde{R}^{ab} \wedge \tilde{R}^{cd} = 0.
\]  

(26)

These equations define the dynamics of the fields at \( B \).

(iii) The effective field theory at \( B \). Equations (24), (25) and (26) can be derived from the remarkable simple four-dimensional action principle

\[
I_B = k \int_B e^\sigma \epsilon_{abcd} \tilde{R}^{ab} \wedge \tilde{R}^{cd},
\]

(27)

where \( \tilde{R}^{ab} \) was defined in Eq. (20). It is direct to see that the variation with respect to “geometric” variables \( e^a \) and \( w^{ab} \) gives rise to Eqs. (24) and (25), while the variation with respect to the dilaton \( \varphi \) gives rise to Eq. (26). \( k \) is a coupling constant with dimension (length)\(^2\).

The action (27) can be thought of as a dilaton-like generalization of the MacDowell-Mansouri [16] action. Note, however, that due to the presence of the dilaton, the Gauss-Bonnet term in Eq. (27) does contribute to the equations of motion. This is similar to what happens in 1+1 dimensions where the dilaton field is included in order to provide a nontrivial dynamics for the two-dimensional Einstein density. As a consequence, the variation of \( I_B \) with respect to the spin connection [Eq. (25)] does not imply the vanishing of the torsion tensor; rather, it gives a differential equation for the spin connection which, probably, means that this theory has a dynamical torsion. A thorough analysis of the solutions of Eq. (25) is beyond the scope of this work. We only mention here that there exists a sector in the theory for which the torsion is equal to zero. That sector is simply obtained by considering the particular class of solutions for which the dilaton is constant,

\[
\varphi = \varphi_0,
\]

(28)

which together with Eq. (25) imply \( T^a = 0 \). We shall see below that matter can be consistently coupled to the four-dimensional theory, keeping the five-dimensional (5D) dynamics unaltered, only on this sector of phase space.

The action (27) is written in the “string frame” where the dilaton multiplies the whole action. By a conformal transformation, one can pass to the “Einstein frame” in which the Hilbert term decouples from the dilaton. In this transformation there exists an ambiguity due to the first-order character of our formalism. Since in this theory the tetrad and spin connection are varied independently, one can make independent fields redefinitions of them. The variation of the spin connection \( w^{ab} \) under the conformal transformation

\[
e^a \rightarrow \Omega e^a
\]

(29)
is not dictated by the theory. Of course one could use the transformation dictated by the second-order theory. If one does so, the action (27) is mapped into an action with complicated couplings between the dilaton and curvature. The precise formula for the transformed action can be obtained by direct replacement of the formulas found, for example, in [17] for conformal transformations.

Perhaps a more interesting conformal transformation\(^1\) is obtained by simply rescaling the tetrad and leaving the spin connection invariant. If we redefine \( e^a \rightarrow e^{-\varphi/2} e^a \), then the action (27) with \( k = -\sigma \ell^2/2 \) is mapped, onto

\[
I_B' = \int_B e^{\sigma - \varphi/2} \epsilon_{abcd} \tilde{R}^{ab} \wedge \tilde{R}^{cd} - \sigma \ell^2 e^\sigma \epsilon_{abcd} \tilde{R}^{ab} \wedge \tilde{R}^{cd}.
\]

(30)

One can recognize in this action the Hilbert term, the cosmological constant term (coupled to the dilaton), and the Gauss-Bonnet term also coupled to the dilaton. Since we did not transform the spin connection, Eq. (30) has no kinetical term for the dilaton. The action (30), with a kinetical term for the dilaton, appears in the one-loop effective action of heterotic string theory and recently black holes solutions have been studied in [18].

The action (30) has another interesting property: It allows the elimination of the dilaton from its own equation of motion. Indeed, varying Eq. (30) with respect to \( \varphi \) one obtains an algebraic equation which can be solved for \( \varphi \):

\[
e^{-\varphi} = \frac{1}{2} \sqrt{\frac{e^{\mu \nu \rho \lambda} \epsilon_{abcd} \tilde{R}^{ab} \tilde{R}^{cd}}{24 \det(e)}}.
\]

(31)

where \( \det(e) \) is the determinant of the tetrad. Replacing back this value of the dilaton into the action one obtains the rather curious modification of general relativity,

\[
I_{\text{red}} = \frac{1}{2} \int_B \left[ e^{\mu \nu \rho \lambda} \epsilon_{abcd} \tilde{R}^{ab} \tilde{R}^{cd} \epsilon_{\mu \nu \rho \lambda} - 2 \sigma \sqrt{6 \det(e)} e^{\mu \nu \rho \lambda} \epsilon_{abcd} \tilde{R}^{ab} \tilde{R}^{cd} \right] d\lambda^4.
\]

(32)

Note that the first term in Eq. (32) is the usual Einstein-Hilbert Lagrangian while the second term is the squared root of the Gauss-Bonnet density. Note also that the second term transforms correctly because both \( \det(e) \) and \( e^{\mu \nu \rho \lambda} \) are densities.

Despite the fact that Eq. (32) is nonlinear in the curvature and is not even polynomial, it still gives rise to first-order equations for the spin connection and tetrad. Indeed, the equations derived from Eq. (32) are totally equivalent to the equations derived from Eq. (30). In general, any function of the curvature and spin connection will give rise to first order equations for \( w^{ab} \) and \( e^a \) provides one uses the Palatini formalism. The price to pay is that the equations of motion will not imply the vanishing of the torsion tensor. Indeed, in nonlinear theories the Palatini and second-order formalisms are not equivalent.

(iv) Coupling matter to the 4D theory. The structure of the five-dimensional equations of motion (3) and (4) suggests a natural way to couple a four-dimensional matter Lagrangian to the dilatonic gravity at \( B \). By virtue of the Bianchi identity, \( \nabla \tilde{R}^{AB} = 0 \), Eq. (4) can be trivially integrated once, obtaining

\[
\epsilon_{ABCD} \tilde{R}^{AB} \wedge e^C = S_{DE},
\]

(33)

where \( S_{DE} \) is a three-form integration function satisfying

\[
\nabla S_{AB} = 0.
\]

(34)

\(^1\)We thank C. Martínez for help on this point.
The consistency of Eq. (33) with Eq. (3) gives another condition for \( S_{AB} \) that will be exhibited below.

The three-form \( S_{AB} \) is an ‘‘integration constant’’ (covariantly constant) of the five-dimensional equations of motion and it represents part of the degrees of freedom of the theory. When projected to \( B \), Eq. (33) is a four-dimensional equation which looks like the Einstein equations with a nonzero energy momentum tensor \( S_{AB} \) and Eq. (34) its associated matter conservation equation. However, under our boundary conditions [see Eq. (11)], we see that the projection of \( S_{AB} \) to \( B \) has been imposed to vanish. In order to be able to prescribe a nonzero value for \( S_{AB} \) at \( B \), and thus find dilaton gravity with matter, we add a boundary term to the 5D action. Consider the action

\[
I = \sum_M \epsilon_{ABCDE} \tilde{R}^{AB}_{\Lambda} \wedge \tilde{R}^{CD}_{\Lambda} \wedge e^{E} + \int_B L^m(W^{AB}, \phi),
\]  

(35)

where \( L^m(W, \phi) \), the 4D matter Lagrangian, is a given functional defined at \( B \). The matter fields will be collectively denoted by \( \phi \). Note that we have coupled the matter fields only to \( W^{AB} \) and not to the vierbein. The reason is that \( e^A \) enters without derivatives in the five-dimensional action and therefore its variation does not couple to the fields at \( B \). Note however that after imposing conditions (19) at \( B \), \( W^{AB} \) contains the four-dimensional tetrad through \( W^a = e^a/l \).

We vary the action (35) with respect to all the fields. All the volume equations of motion remain the same because we have only added a boundary term to the action. The only modification which is required is the boundary equation (12) which now reads

\[
\epsilon_{ABCDE} \epsilon^{\mu\nu\lambda\rho} R^{AB}_{\mu\nu} R^{CD}_{\lambda\rho} e^C = S^D_{DE},
\]  

(36)

where the ‘‘energy-momentum’’ tensor is defined by

\[
S^\mu_{AB} = -\frac{\partial L^m(W, \phi)}{\partial W^{AB}_\mu}.
\]  

(37)

The equations of motion for the matter follow from the variation of \( L^m \) with respect to \( \phi \). Note that since the boundaries of \( B \) are the two disconnected ‘‘rings’’ defined by the intersection of \( \Sigma_1 \) and \( \Sigma_2 \) with \( B \) (see Fig. 1), it is enough to prescribe initial and final values for the matter fields in order cancel all boundary terms coming from the variation of the matter Lagrangian.

We now study the compatibility of Eq. (36) with the bulk equations of motion projected to \( B \), namely Eqs. (13) and (14). Compatibility of Eqs. (36) with Eq. (14) imposes

\[
\nabla_\mu S^\mu_{AB} = 0,
\]  

(38)

while compatibility of Eq. (36) with Eq. (13) imposes

\[
R^{AB}_{\mu\nu} S^\nu_{AB} = 0.
\]  

(39)

Condition (38) follows directly from taking the derivative of Eq. (36) and comparing with Eq. (14). Condition (39) is obtained by multiplying Eq. (36) by \( R_{DE}^{\rho\sigma} \), using some simple combinatorial identities and comparing with Eq. (13). The geometrical meaning of Eqs. (38) and (39) is straightforward and it provides a remarkable link between the five-dimensional equations of motion and the symmetries of the four-dimensional matter Lagrangian. Indeed, Eq. (38) is associated with invariance of \( L^m(W, \phi) \) under local gauge transformations

\[
\delta W^{AB} = -\nabla_\mu \lambda^{AB}_\mu,
\]  

(40)

where \( \lambda^{AB}_\mu \) is an arbitrary parameter. Equation (39), on the other hand, is associated with invariance of \( L^m(W, \phi) \) under (improved [14]) diffeomorphisms

\[
\delta_\xi W^{AB} = R^{AB}_{\mu\nu} \xi^\nu,
\]  

(41)

where \( \xi^\mu \) is an arbitrary vector field at \( B \). Thus, compatibility between the boundary condition (36) and the five-dimensional equations of motion is ensured provided the matter Lagrangian is invariant under local gauge transformations and diffeomorphisms.

Invariance under diffeomorphisms is the minimum symmetry that any Lagrangian coupled to gravity should possess. Hence, Eq. (39) will be satisfied by any reasonable choice of \( L^m \). Actually, we shall see shortly that for torsionless configurations, Eq. (39) is identically satisfied. That is the reason that Eq. (39) is not commonly found in the literature.

The invariance of \( L^m \) under the gauge transformations (40) is more subtle. The gauge transformation (40) is not a simple Lorentz rotation; rather, it implies a full local five-dimensional SO(3,2) symmetry [or SO(4,1) depending on the sign of \( \sigma \)]. Even though one can construct some toy examples of matter Lagrangians possessing such symmetry, most of the phenomenological and commonly used forms of matter are not SO(3,2) invariant but only invariant under the subgroup of Lorentz transformations SO(3,1). The problem faced here can be put in a different, perhaps more transparent, way. As we saw in the vacuum case, in order to find a sensible theory at \( B \) we need to impose the conditions (19). The equations derived from Eq. (36) after imposing conditions (19) are

\[
\epsilon_{abcd} \tilde{R}^{ab}_{\cd} \wedge e^c = S_d,
\]  

(42)

\[
\epsilon_{abcd} (2 \sigma T^a \wedge e^b - i^2 \tilde{R}^{ab} \wedge d \varphi) = i S_{cd},
\]  

(43)

where we have defined

\[
S_a = -\sigma \frac{\partial L^m}{\partial e^a}, \quad S_{ab} = -\frac{\partial L^m}{\partial W^{ab}}.
\]  

(44)

Note that these equations are just the generalizations of Eqs. (24) and (25) with matter. As in the vacuum case, the fifth component of Eq. (13) contributes in a nontrivial way to the dynamics at \( B \). Hence we add the equation

\[
\epsilon_{abcd} \tilde{R}^{ab}_{\cd} \wedge \tilde{R}^{cd} = 0,
\]  

(45)

which, together with Eqs. (42) and (43), defines the field theory at \( B \).

Equations (42) and (43) are Lorentz and diffeomorphism invariant, but they do not possess the larger SO(3,2) [SO(4,1)] local symmetry. Therefore the symmetries of the left-hand sides of Eqs. (42) and (43) are not enough to ensure
Eq. (38). There exists, however, a large class of solutions of the equations of motion which do satisfy Eqs. (38) and (39). Consider the case of spinless matter ($S_{ab} = 0$) and solutions for which the torsion tensor is equal to zero ($T^a = 0$). Under this restriction both Eqs. (38) and (39) are identically satisfied. Indeed, Eq. (39) is equivalent to

$$\tilde{R}^{ab}_{\mu
u} S_{ab} + 2 T_{\mu\nu} S_a^a = 0,$$

(46)

which is identically satisfied for $T^a = 0 = S_{ab}$. On the other hand, Eq. (38) written in the SO(3,1) language becomes

$$(\sigma l) D_{\mu} S_{ab}^{\mu} + S_{ab}^{\mu} e_{\mu}^{b} = 0,$$

(47)

$$(\sigma l) D_{\mu} S_{ab}^{\mu} - e_{\mu b} S^{\mu}_{a} + e_{\mu b} S_{a}^{\mu} = 0.$$

(48)

For $S_{ab} = 0$, these equations represent diffeomorphism and Lorentz invariance of Eqs. (42) and (43). Indeed, Eq. (47) is equivalent to the conservation of the energy-momentum tensor $T^{\mu \nu} = S^{ab} e_a^\nu$, while Eq. (48) imposes $T^{\mu \nu} = T^{a \mu}$.

A different point of view for the same issue is provided by the study of the transformation for the tetrad induced by Eq. (40):

$$\delta e_a^\mu = - D_{\mu} \lambda^a + \lambda_a^{b} e_{\mu}^{b},$$

(49)

where $\lambda^{AB} = (\lambda^{ab}, (\sigma l) \lambda^a)$. The second term in Eq. (49) is just a local Lorentz rotation which is in fact a symmetry of the equations of motion (42) and (43). The first term is a symmetry of the equations only in the case when the torsion tensor is zero. In that case, $\delta e_a^\mu = - D_{\mu} \lambda^a$ represents a diffeomorphism with a parameter $\xi^\mu = e_a^\mu \lambda^a$. Note that for spinless matter ($S_{ab} = 0$) the matter Lagrangian is independent of the spin connection and therefore the transformation induced by Eq. (40) is irrelevant.

In summary, we have seen that the dynamics at the boundary can be altered by the adding of a boundary matter Lagrangian without modifying the 5D equations only in the sector for which the torsion is zero. It is rather disturbing that we have been forced to set the torsion tensor equal to zero at $B$ rather than deriving it from the equations of motion. The relevant question, which will not be addressed here, is whether setting $T^a = 0$ destroys any degrees of freedom. A less harmful way to set $T^a = 0$ is by considering the class of solutions for which the dilaton is constant. For a constant dilaton, which can thus be identified with Newton’s constant, Eqs. (42) and (43) are exactly the Einstein equations with matter and for $S_{ab} = 0$ Eq. (43) implies $T^a = 0$.

(v) Anti–de Sitter Chern-Simons gravity. The reader familiar with CS gravity may wonder if the above considerations can be applied to the (anti–) de Sitter CS theory. The answer to that question is affirmative for (and only for) the anti–de Sitter theory as we now explain.

The (anti-)de Sitter CS theory in five dimensions is defined by the action [11,12]

$$I_L = \frac{1}{2} \epsilon_{ABCDE} \left[ \tilde{R}^{AB} \wedge \tilde{R}^{CD} \wedge e^E \right. \left. - \frac{2\kappa}{3L^2} \tilde{R}^{AB} \wedge e^C \wedge e^D \wedge e^E \right. \left. + \frac{1}{5L^2} e^A \wedge e^B \wedge e^C \wedge e^D \wedge e^E \right],$$

(50)

where, as before, $\eta_{AB} = \text{diag}(-1,1,1,1,1)$ and $\kappa = \pm 1$. Depending on the signs of $\sigma$ and $\kappa$ the action (50) can be shown to be a CS action for the groups [11]

SO(5,1) if $\sigma = \kappa = 1$ (de Sitter),

SO(4,2) if $\sigma = - \kappa$ (anti–de Sitter),

SO(3,3) if $\sigma = \kappa = - 1$.

In Eq. (50) the term linear in the curvature is the 5D Hilbert term, while the last term is the five-dimensional cosmological constant, parametrized by $L$. The action (50) with $\sigma = 1$ has a sensible interpretation as a 5D gravitational action, and in the case $\kappa = 1$ black holes solutions exist [12]. Also, it can be proved that the above action does possess local degrees of freedom [8].

We shall now study the problem of boundary terms and boundary conditions arising in the variation of Eq. (50). We vary Eq. (50) with respect to all the fields obtaining the bulk equations of motion

$$\epsilon_{ABCDE} F_{AB} \wedge F_{CD} = 0,$$

(52)

$$\epsilon_{ABCDE} F_{AB} \wedge \tilde{\Omega}_{A} = 0,$$

(53)

where $\tilde{\Omega}_{A} = \nabla e^A$ and we have defined

$$F_{AB} = \tilde{R}^{AB} - \frac{\kappa}{L^2} e^A \wedge e^B.$$

(54)

We also obtain the equation at $B$,

$$\epsilon_{ABCDE} \left( F_{AB} \wedge e^C + \frac{2\kappa}{3L^2} e^A \wedge e^B \wedge e^C \right) = 0,$$

(55)

ensuring that all boundary terms at $B$ are equal to zero. We stress that while Eqs. (52) and (53) are five-dimensional equations, Eq. (55) is a four-dimensional equation defined only at $B$. It is worth noticing the similarity between Eqs. (52) and (53) with their Poincaré counterparts, Eqs. (3) and (4). Equations (52) and (53) are obtained from Eqs. (3) and (4) just by replacing $\tilde{R}^{AB}$ by $F_{AB}$. The equation at the boundary, on the other hand, does not enjoy the same property [see Eqs. (11) and (55)]. We shall see shortly that the extra term appearing in Eq. (55) is necessary for compatibility between Eq. (55) and the projections to $B$ of Eqs. (52) and (53).

As we did in Poincaré theory, we need to study the compatibility of the boundary condition (55) with the bulk equations of motion. The procedure is very much like in Poincaré theory, and so we shall only quote the main results. In the following, we only consider the projections of Eqs. (52) and (53) to $B$ which, in differential forms notation, look the
same, and so we do not write them again. We first note that Eq. (53) (projected to \( B \)) is a direct consequence of Eq. (55). Indeed, taking the covariant (\( \nabla \)) derivative of Eq. (55) we get Eq. (53). Hence, any solution satisfying Eq. (55) will automatically satisfy Eq. (53) at \( B \). Equation (52) is more complicated. Just as in the Poincaré case, only part of Eq. (52) is satisfied once Eq. (55) is imposed. Again, to explicitly isolate the part of Eq. (52) not contained in Eq. (55), we break the explicit five-dimensional local symmetry as in Eq. (16) and then we impose conditions (19) in order to have the same number of fields and equations at \( B \). A special feature of the (anti–)de Sitter theory is that the consistency of the boundary condition Eq. (55) with Eq. (52) fixes the parameter \( l \) appearing in condition (19) in terms of the parameter \( L \) appearing in Eq. (50) by \( l^2 = L^2 \). Also, the signatures \( \sigma \) and \( \kappa \) will be related by \( \sigma = - \kappa \). Thus, according to Eq. (51) only the anti–de Sitter [SO(4,2)] theory admits the boundary condition Eq. (55). We note that the action (50) admits black holes solutions only in this case [12].

A quick way to obtain these restrictions on the parameters \( l \) and \( \sigma \) is by considering Eq. (52) (projected to \( B \)) in terms of its SO(3,1) components

\[
\epsilon_{abcd} F^{ab} \wedge F^{cd} = 0, \quad E = e, \quad \sigma = - \kappa, \quad l^2 = L^2. \tag{60}
\]

The compatibility of Eq. (56) with Eq. (55) can be checked using Eq. (53), which we already know is consistent with Eq. (55). Indeed, the component \( E = 5 \) of Eq. (53) reads

\[
\epsilon_{abcd} F^{ab} \wedge \tilde{F}^e = 0. \tag{58}
\]

Therefore, comparing Eq. (56) with Eq. (58) we see that they become identical if \( F^{a5} = f T^a \) where \( f \) is any smooth function. By direct application of Eqs. (21), (22), and (54) we find that \( F^{a5} = f T^a \) implies

\[
\sigma = - \kappa, \quad l^2 = L^2.
\]

In complete analogy with the Poincaré case, Eq. (57) is not a consequence of the boundary conditions (55) and therefore it contributes in a nontrivial way to the dynamics at \( B \). Thus, the theory at \( B \) is described by Eqs. (55) plus (57). Note, however, that conditions (60) imply that \( F^{a5} = R^{a5} \) and thus Eq. (57) becomes simply

\[
\epsilon_{abcd} R^{ab} \wedge R^{cd} = 0. \tag{61}
\]

The equations at \( B \) can again be derived from a simple four-dimensional action principle:

\[
I = \int_B \epsilon_{abcd} \left( R^{ab} \wedge e^c \wedge e^d + \frac{\kappa}{3L^2} e^a \wedge e^b \wedge e^c \wedge e^d - \frac{\kappa L^2}{2} R^{ab} \wedge R^{cd} \varphi \right), \tag{62}
\]

in which the dilaton is coupled to the Gauss-Bonnet term only. Note that the first and second terms are the usual Einstein-Hilbert and cosmological terms.

The variation of Eq. (62) with respect to \( \varphi \) gives Eq. (61), while the variation with respect to the “geometrical” variables \( w^{ab} \) and \( e^a \) gives,

\[
\epsilon_{abcd} (2 T^a \wedge e^b - \kappa L^2 R^{ab} \wedge d \varphi) = 0, \tag{63}
\]

\[
\epsilon_{abcd} \left( R^{ab} \wedge e^c + \frac{2 \kappa}{3L^2} e^a \wedge e^b \wedge e^c \right) = 0. \tag{64}
\]

These equations are exactly equivalent to Eq. (55) written in SO(3,1) components, after conditions (19) are imposed. The action (62) can also be interpreted as a dilatoniclike gravity. Note that the coupling of the dilaton with the gravitational variables is different from the Poincaré case.

(vi) Other boundary conditions. Finally, we would like to consider other possible boundary conditions. Let us go back to Eq. (9) and study a different method to treat the boundary terms; namely, we shall now impose boundary conditions such that Eq. (9) can be written as a total variation.

The boundary term (9) is clearly not a total variation unless some boundary conditions linking \( e^A \) with \( W^{AB} \) are imposed. One could also use boundary conditions for which \( e^A \) is fixed at \( B \). However, those boundary conditions may overdetermine the Cauchy problem and also the propagating amplitudes would not depend on initial and final data but also on intermediate data defined at \( B \). To avoid the need of giving data at \( B \), the standard procedure is to set some fields equal to a fixed, not arbitrary, given value [e.g., asymptotic flatness in general relativity (GR)].

We thus face the problem of imposing boundary conditions such that Eq. (9) can be written as a total variation. It is remarkable that precisely conditions (19) provide such conditions for a fixed value of \( \varphi \). For simplicity, we fix \( \varphi = \varphi_0 \) which implies \( e^5 = 0 \). It is easy to check that under conditions (19) one has the identity

\[
\int_B \epsilon_{ABCD} \tilde{R}^{AB} \wedge e^C \wedge \delta W^{DE} = \delta \left[ \frac{\sigma}{2L} \int_B \epsilon_{abcd} \left( 2 R^{ab} - \frac{\kappa}{3L^2} e^a \wedge e^b \right) \wedge e^c \wedge e^d \right]. \tag{65}
\]

(This identity involves an integration by parts in the right-hand side which generates a boundary term at \( \partial B \). Since the fields at \( \partial B \) are fixed, this boundary term is equal to zero.)

We thus consider the improved five-dimensional action
\[
I = \frac{1}{2} \int_M \varepsilon_{ABCD} \tilde{R}^{AB} \wedge \tilde{R}^{CD} \wedge e^E \\
+ \frac{\sigma}{2l} \int_B \varepsilon_{abcd} \left( 2R^{ab} - \frac{\sigma}{l^2} e^a \wedge e^b \right) \wedge e^c \wedge e^d, \tag{66}
\]

which has well-defined functional derivatives under variations satisfying condition (19) at \(B\). Recall that now Eq. (11) is not imposed. Note that the boundary term that we have added is just the Einstein-Hilbert action with a cosmological constant.

The variation of the action (66) gives rise to Eqs. (4) and (3) which together with the boundary conditions completely define the theory. Since we have not imposed Eq. (11) at \(B\), we do not find a “boundary theory” in this case. Despite the presence of the Einstein-Hilbert action in Eq. (66), the equations of motion at \(B\) are the projections to \(B\) of Eqs. (4) and (3) which are not equivalent to the Einstein equations. However, one can speculate that quantum mechanically the action (66) induces general relativity at \(B\), at least in a saddle point approximation of the path integral. This can be seen as follows.

A key property of the CS action is that the value of the bulk term evaluated on any solution of the equation of motion is equal to zero. Therefore, the value of \(I\) given in Eq. (66) evaluated on any solution of the equations of motion is equal to the boundary term, that is, equal to the Einstein-Hilbert action at \(B\). Since in the CS variational principle we have only imposed conditions (19) with \(\varphi = \varphi_0\), we have to vary the boundary action with respect to the remaining fields, that is, \(e^a\) and \(w^{ab}\), thus obtaining general relativity at \(B\).

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