Constant curvature black holes

Máximo Bañados
Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago, Chile
and Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile
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Constant curvature black holes are constructed by identifying points in anti–de Sitter space. In n dimensions, the resulting topology is $\Re^{n-1}\times S_1$, as opposed to the usual $\Re^2\times S_{n-2}$ Schwarzschild black hole, and the corresponding causal structure is displayed by an $(n-1)$-dimensional picture, as opposed to the usual two-dimensional Kruskal diagram. The five-dimensional case, which can be embedded in a Chern-Simons supergravity theory, is analyzed in detail. [S0556-2821(97)02924-X]

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The family of black objects has grown considerably during the last decade. Black holes, black strings, and black branes have been studied in various dimensions and theories, and it is now commonly believed that they play an important role in string theory—the most promising theory to unify gravity with the other fundamental forces.

Black holes, black branes, and black strings have event horizons, a null surface in spacetime beyond which light cannot escape. This property is best displayed in terms of the causal structure. The standard Kruskal picture (without charge or angular momentum) is drawn in Fig. 1. The curved lines represent the future and past singularities and the diagonals the horizon. A future-directed observer in region II will necessarily hit the singularity since in order to go back to region I which is connected to infinity, he or she would need a velocity greater than light. In the simplest situation, each point in Fig. 1 represents a $(n-2)$ sphere with n the dimension of spacetime. The spacetime topology is thus $\Re^2\times S_{n-2}$.

In this paper we shall construct metrics with constant negative curvature whose Kruskal diagram will be given by the picture shown in Fig. 2. The hyperboloid represents the singularity, while the cone represents the horizon. Drawing the two surfaces together one obtains a picture equivalent to Fig. 1 rotated around the z axis. An observer falling into the region in between the hyperboloid and the cone cannot escape back because, as before, that would require a velocity greater than light. In that sense, Fig. 2 displays a black hole.

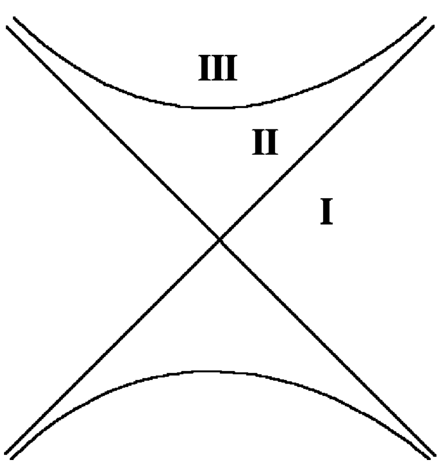

FIG. 1. Standard Kruskal diagram. Region I is connected to infinity and region II is the interior of the black hole.

In our construction, each point in Fig. 2 will represent a circle and the spacetime topology is thus $\Re^3\times S_1$. In an arbitrary dimension n, Fig. 2 will be replaced by its natural $n-1$ generalization and the black hole will then have the topology $\Re^{n-1}\times S_1$.

The existence of a ''$\Re^{n-1}\times S_1$ black hole'' was first suggested in [1], and also extensively discussed with Henneaux. The main two problems faced on that occasion, which are addressed here, were an apparent change of signature when crossing the horizon and the definition of conserved charges associated to asymptotic symmetries.

The construction of the $\Re^{n-1}\times S_1$ black hole is a natural extension of the procedure yielding the 2+1 black hole from anti–de Sitter space with identified points [2,3]. We shall then start by briefly reviewing that construction [4], in the nonrotating case.

In three dimensions, anti–de Sitter space is defined as the universal covering of the surface

$$-x_0^2+x_1^2+x_2^2-x_3^2=-l^2. \tag{1}$$

This surface has six Killing vectors, two rotations, and four boosts. Pick up the Killing vector $\xi=(r_+/l)(x_2\partial_3+x_3\partial_2)$ whose norm $\xi^2=(r_+^2/l^2)(-x_2^2+x_3^2)$ can be negative, zero, or positive (r_+ is an arbitrary real number). The surface (1) can be rewritten as

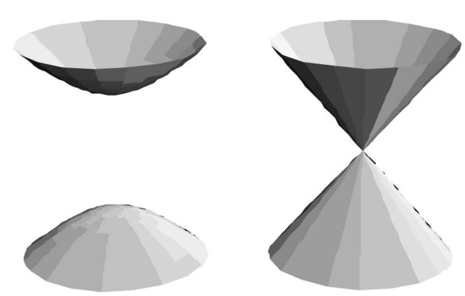

FIG. 2. The hyperboloid represents the singularity and the hypercone the horizon. Drawing the two surfaces together gives Fig. 1 rotated around the z axis.

$$x_0^2=x_1^2+l^2(1-\xi^2/r_+^2), \tag{2}$$

and can be plotted parametrically in terms of the values of ξ^2. One finds that the semiplane $x_0|x_1$, $x^0+x^1>0$, has three regions separated by $\xi^2=0$ and $\xi^2=r_+^2$. Indeed, the picture is the same as in Fig. 1 on which $\xi^2=0$ represents the singularity and $\xi^2=r_+^2$ the horizon:

$$\text{I:}\quad r_+^2<\xi^2<\infty,$$

$$\text{II:}\quad 0<\xi^2<r_+^2,$$

$$\text{III:}\quad -\infty<\xi^2\leqslant 0.$$

Also, each point in Fig. 1 represents a one-dimensional noncompact manifold: the values of x_2 and x_3 with $x_2^2-x_3^2$ (the norm of the Killing vector) fixed. (Regions I, II, and III are repeated in the lower semiplane $x^0+x^1<0$.)

This, of course, does not transform anti–de Sitter space into a black hole. What does produce the black hole is the identification of points along the orbit of ξ. Since ξ is a Killing vector one can produce a new metric with constant curvature by taking the quotient of the surface (1) with a discrete subgroup generated by ξ. If we do so, region III ($\xi^2\leqslant 0$) acquires a pathological chronological structure and therefore it must be cut off from the physical spacetime [4]. In that sense, the surface $\xi^2=0$ becomes a singularity. Moreover, the noncompact one-dimensional manifold orthogonal to Fig. 1 becomes isomorphic to S_1. The quotient space has thus the topology $\mathfrak{R}^2\times S_1$ and the induced metric is the $2+1$ black hole [4].

The above picture has a natural generalization to higher dimensions [2,3]. In n dimensions anti–de Sitter space is defined as (the universal covering) of the surface

$$-x_0^2+x_1^2+\cdots+x_{n-2}^2+x_{n-1}^2-x_n^2=-l^2. \tag{3}$$

Consider the boost $\xi=(r_+/l)(x_{n-1}\partial_n+x_n\partial_{n-1})$ with norm $\xi^2=(r_+^2/l^2)(-x_{n-1}^2+x_n^2)$. As before, we plot parametrically the surface (3) in terms of the values of ξ^2. The horizon ($\xi^2=r_+^2$) is now represented by the hypercone

$$x_0^2=x_1^2+\cdots+x_{n-2}^2, \tag{4}$$

while the singularity ($\xi^2=0$) is represented by the hyperboloid

$$x_0^2=x_1^2+\cdots+x_{n-2}^2+l^2, \tag{5}$$

just as illustrated in Fig. 2. We now identify points along the orbit of ξ obtaining the desired causal structure. The region behind the hyperboloid ($\xi^2<0$) has to be removed from the physical spacetime because it contains closed timelike curves. The hyperboloid is thus a singularity because timelike geodesics end there. Again, each point in Fig. 2 represents a circle (the identified line) and the topology of the quotient space is $\mathfrak{R}^{n-1}\times S_1$.

To go further in the discussion, let us introduce local coordinates on anti–de Sitter space (in the region $\xi^2>0$) adapted to the Killing vector used to make the identifications. We introduce the n dimensionless local coordinates (y_α,ϕ) by

$$x_\alpha=\frac{2ly_\alpha}{1-y^2},\quad \alpha=0,\ldots,n-2,$$

$$x_{n-1}=\frac{lr}{r_+}\sinh\left(\frac{r_+\phi}{l}\right), \tag{6}$$

$$x_n=\frac{lr}{r_+}\cosh\left(\frac{r_+\phi}{l}\right),$$

with

$$r=r_+\frac{1+y^2}{1-y^2} \tag{7}$$

and $y^2=\eta_{\alpha\beta}y^\alpha y^\beta$ [$\eta_{\alpha\beta}=\mathrm{diag}(-1,1,\ldots,1)$]. The coordinate ranges are $-\infty<\phi<\infty$ and $-\infty<y^\alpha<\infty$ with the restriction $-1<y^2<1$.

The induced metric has the Kruskal form

$$ds^2=\frac{l^2(r+r_+)^2}{r_+^2}dy^\alpha dy^\beta\eta_{\alpha\beta}+r^2d\phi^2, \tag{8}$$

and the Killing vector reads $\xi=\partial_\phi$ with $\xi^2=r^2$. The quotient space is thus simply obtained by identifying $\phi\sim\phi+2\pi n$, and the resulting topology is $\mathfrak{R}^{n-1}\times S_1$. With the help of Eq. (6), it is clear that the Kruskal diagram associated to this geometry is the one shown in Fig. 2 extended to an arbitrary dimension n. Thus, the metric (8) represents the $\mathfrak{R}^{n-1}\times S_1$ black hole written in Kruskal coordinates. Note also that the above metric is a natural generalization of the $2+1$ black hole. Indeed, setting $n=3$ in Eq. (8) gives the nonrotating $2+1$ black hole metric written in Kruskal coordinates [4].

Hereafter we shall restrict the discussion to the five dimensional case which has some special features that will be explained below. One may wonder if there exists Schwarzschild coordinates for the above metric. We shall see that they exist only in the exterior region. In particular one cannot find Schwarzschild coordinates interpolating the inner and outer regions.

Let us introduce local ''spherical'' coordinates (t,θ,χ,r) in the hyperplane y^α:

$$y_0=f\cos\theta\sinh(r_+t/l),\quad y_2=f\sin\theta\sin\chi,$$

$$y_1=f\cos\theta\cosh(r_+t/l),\quad y_3=f\sin\theta\cos\chi,$$

with $f(r)=[(r-r_+)/(r+r_+)]^{1/2}$. (Note that these coordinates, with ranges $0<\theta<\pi$, $0\leqslant\chi<2\pi$, $-\infty<t<\infty$, and $r_+<r<\infty$, do not cover the whole region $r>r_+$ but only $-1<y_2,y_3<1$.) The metric (8) acquires the Schwarzschild form

$$ds^2 = l^2 N^2 d\Omega_3 + N^{-2} dr^2 + r^2 d\phi^2, \tag{9}$$

with $N^2(r) = (r^2 - r_+^2)/l^2$ and

$$d\Omega_3 = -\cos^2\theta dt^2 + \frac{l^2}{r_+^2}(d\theta^2 + \sin^2\theta d\chi^2). \tag{10}$$

The horizon in these coordinates is located at $r = r_+$, the point where N^2 vanishes. It should now be clear why the Kruskal diagram is four dimensional (in five dimensions): the ''lapse'' function N^2 multiplies not a single coordinate but the whole (hyperbolic) three sphere $d\Omega_3$. It is also clear that one cannot find Schwarzschild coordinates interpolating the outer and inner regions. For $r < r_+$, the metric (9) changes its signature and therefore it does not represent the interior of the black hole.

The black hole just constructed has a Euclidean sector which can be obtained by setting $\tau = it$ in Eq. (9), or $y_0 \to iy_0$ in Eq. (8). In this sector the coordinates (t, θ, χ, r) do cover the whole Euclidean black hole spacetime which, as usual, is isomorphic to the exterior of the Minkowskian black hole.

For later use we point out that the boundary of the spatial sections has the topology $S_2 \times S_1$. This is easily seen by setting $t, r = \text{const}$ in Eq. (9) or, alternatively, by setting $y_0, r = \text{const}$ in Eq. (8).

Just as in $2+1$ dimensions, angular momentum in the plane $t|\phi$ can be added by considering a different Killing vector to do the identifications. Here, we shall not carry the complete geometrical construction which for dimensions greater than three is not trivial. Instead, we shall add a new charge by boosting the above metric in the plane $t|\phi$. This is most easily done by setting $r_+ = l$ in Eq. (9), making the replacements

$$t \to \beta t \frac{r_+}{l^2} + (\phi - \Omega\beta t)\frac{r_-}{l}, \tag{11}$$

$$\phi \to \beta t \frac{r_-}{l^2} + (\phi - \Omega\beta t)\frac{r_+}{l} \tag{12}$$

($r_+ > r_-$ arbitrary constants), and identifying points along the new angular coordinate ϕ: $\phi \sim \phi + 2\pi n$. The parameters β and Ω are introduced here only to make the canonical structure of the global charges explicit: β is the conjugate to the energy while Ω is the conjugate to the angular momentum. One could set $\beta = l$ and $\Omega = 0$ without altering the physical properties of the solution at all.

The explicit form of the resulting metric is not very illuminating so we do not include it here. We only point out that the constant r_+ parametrizes the location of the outer horizon, and that the new metric has two independent conserved charges (see below). In the Euclidean formalism, the time coordinate $\tau = -it$ must be periodic in order to avoid conical singularities. This gives the value $\beta = (2\pi r_+ l^2)/(r_+^2 - r_-^2)$ (with $0 \leq t < 1$) which can be interpreted as the inverse temperature of the black hole.

Since the above geometries are locally anti–de Sitter, they are natural solutions of Einstein equations with a negative cosmological constant. However, due to the nonstandard asymptotic behavior of Eq. (9) one finds that all conserved charges are infinite [5]. This is a serious obstacle since if no physical conserved charges can be defined the physical relevance of the solution is not clear.

We would like to point out here that global conserved charges associated to these black holes can be defined in the context of a Chern-Simons supergravity theory in five dimensions purposed sometime ago by Chamseddine [6]. This action is constructed as a Chern-Simons theory for the supergroup SU(2,2$|N$) [6] and it represents a natural extension of the three-dimensional supergravity theory constructed in [7]. The explicit form of the action is not needed here (it can be found in [6]). What is more useful are the equations of motion which read (setting all fermions and non-Abelian gauge fields equal to zero)

$$l\epsilon_{abcde}\tilde{R}^{ab}\wedge\tilde{R}^{cd} - 2F\wedge T_e = 0, \tag{13}$$

$$2\epsilon_{abcde}\tilde{R}^{ab}\wedge T^c + lF\wedge\tilde{R}_{de} = 0, \tag{14}$$

$$\tilde{R}^{ab}\wedge\tilde{R}_{ab} - 2l^{-2}T^a\wedge T_a + \Delta F\wedge F = 0, \tag{15}$$

where $\Delta = N^{-4} - 4^{-4}$ and N is the number of fermions in the action. $T^a = De^a$ is the two-form torsion, and

$$\tilde{R}^{ab} = R^{ab} + l^{-2}e^a\wedge e^b, \tag{16}$$

where $R^{ab} = dw^{ab} + w^a{}_c\wedge w^{cb}$ is the two-form Lorentz curvature. Finally, the two-form $F = dA$ is the curvature associated to the U(1) gauge field A which is necessary to achieve supersymmetry. The above equations of motion are not equivalent to the five-dimensional Einstein equations. However, note that in the sector $F = T^a = 0$ and small curvatures, $R^2 \approx 0$, they do reduce to Einstein equations.

Note that the group SU(2,2$|N$) has as a subgroup SO(4,2)$\times$U(1). In fact, the above equations can be interpreted as a Chern-Simons theory for the group SO(4,2)$\times$U(1). Also, since SO(4,2) is isomorphic to the anti–de Sitter group in five dimensions (whose Lie algebra is generated by J_{ab} and P_a), $\tilde{R}^{ab}$ and T^a are, respectively, the projections of the SO(4,2) curvature along J_{ab} and P_a.

The dynamics of Chern-Simons theories has been studied in general in [8]. In particular, it was shown in [8] that those theories with a gauge group $G\times$U(1) enjoy a drastic simplification in the canonical and asymptotic structure if (i) the last term in Eq. (15) is identically zero and (ii) F has maximum rank [8]. Condition (i) is a restriction on the invariant tensor necessary to construct the Chern-Simons theory and (ii) ensures the existence of local degrees of freedom. We shall then consider here the theory with $N = 4$ $(\Delta = 0)$ and study solutions for which F has rank 4 (in five dimensions, the maximum rank of a two-form is 4).

It can now be seen that the geometries described at the beginning of this note solve the above equations of motion (for $N = 4$) because they have constant curvature ($\tilde{R}^{ab} = 0$) and zero torsion ($T^a = 0$). Note also that F is left arbitrary and therefore it can have maximum rank.

Global conserved charges associated to the above equations are easily constructed [8]. Let $E=E^a P_a+(1/2)E^{ab}J_{ab}$ be the left-hand side of Eq. (13) and (14), and let $\delta e^a, \delta w^{ab}$ be perturbations of the vielbein and spin connection. It follows that δE is a covariantly conserved current of the classical history. Now, let $\lambda=\lambda^a P_a+(1/2)\lambda^{ab}J_{ab}$ be a Killing vector of the background configuration ($D\lambda=0$ where D is the anti–de Sitter covariant derivative), then $\delta J=Tr(\lambda\,\delta E)$ is a conserved current in the ordinary sense. As in any gauge theory, δJ is a total derivative and integrating it over a spatial section Σ with boundary $\partial\Sigma$ one obtains

$$\delta Q[\eta_a,\eta_{ab}]=\frac{1}{8\pi^2}\int_{\partial\Sigma}F\wedge(2\,\eta_a\delta e^a-l\,\eta_{ab}\delta w^{ab}). \tag{17}$$

This formula holds under the boundary condition $\widetilde{R}^{ab}=0=T^a$ which, of course, is satisfied by our solution. The normalization factor $8\pi^2$ comes from the volume of $\partial\Sigma=S_2\times S_1$.

The formula (17) depends on the two-form F which was not determined by the equations of motion. The only local conditions over F are $dF=0$ (since $F=dA$), and that it must have maximum rank [8]. In particular, F must be different from zero everywhere. It turns out that global considerations suggest a natural choice for the pull back of F into $\partial\Sigma=S_2\times S_1$, namely, proportional to the area two-form of S_2, $F=k\ \sin\theta d\theta\wedge d\chi$. Note that $dF=0$ implies that k is constant over S_1.

The formula for the charge thus becomes

$$\delta Q[\eta_a,\eta_{ab}]=\frac{k}{8\pi^2}\int_{\partial\Sigma}(2\,\eta_a\delta e^a_\phi-l\,\eta_{ab}\delta w^{ab}_\phi)dS, \tag{18}$$

where e^a_ϕ and w^{ab}_ϕ are the projections of the veilbein and spin connection along S_1 (parametrized by ϕ), and $dS=\sin\theta d\theta d\chi d\phi$.

The black hole geometries described above have two commuting Killing vectors ∂_t and ∂_ϕ whose conserved charges can be associated, respectively, to the energy and angular momentum of the solution. Due to the flatness of the anti–de Sitter curvature in our solution, a diffeomorphism with parameter ξ^μ is equivalent to an anti–de Sitter gauge transformation with parameter $\xi^\mu[e^a_\mu P_a+(1/2)w^{ab}_\mu J_{ab}]$ [9]. Hence, the Killing displacement ∂_t can be replaced by a gauge transformation with parameter $e^a_t P_a+(1/2)w^{ab}_t J_{ab}$, and ∂_ϕ by a gauge transformation with parameter $e^a_\phi P_a+(1/2)w^{ab}_\phi J_{ab}$.

Computing the energy M and angular momentum J as described above one finds

$$M=\frac{1}{l}Q[e^a_t,w^{ab}_t]=k\frac{2r_+r_-}{l^2}, \tag{19}$$

$$J=Q[e^a_\phi,w^{ab}_\phi]=k\frac{r_+^2+r_-^2}{l}, \tag{20}$$

where we have normalized the time Killing vector by $\beta=l$, and $\Omega=0$ (no shift at infinity). In obtaining M and J we have used the boundary condition $\delta k=0$ [8]. Note that k, which acts as a coupling constant, is not a universal parameter in the action but the fixed value of the U(1) field strength F. This is similar to what happens in string theory where the coupling constant is equal to the value of the dilaton field at infinity.

Comparing the above values for M and J with those obtained in the $2+1$ theory [4], one discovers that they are reversed. One can easily trace back the reason for this interchange between mass and angular momentum: in the action, the only term that contributes to the conserved charges is $A\wedge(l^2\widetilde{R}^{ab}\wedge\widetilde{R}_{ab}-2T^a\wedge T_a)$ which is not the usual Hilbert Lagrangian. This term, which involves the SO(4,2) Pontryagin density as opposed to the Euler density, is not parity invariant unless A is a density. Note also that the black hole horizon exists only for $J\geqslant M$, as opposed to the standard bound $M\geqslant J$ [10]. The permutation of charges is a well-known phenomenon when a duality transformation is applied. In this case, the ''duality transformation'' comes from the fact that the relevant part of the action was constructed with η_{ab} tensors instead of the five-dimensional Levi-Cevita symbol ϵ_{abcde}. It is worth mentioning here that the same situation is observed in $2+1$ dimensions if one considers, instead of the usual Einstein-Hilbert action, the ''exotic'' $2+1$ action [9] which is also constructed using η_{ab} tensors instead of the Levi Civita symbol. Another example where this phenomena occurs was reported in [11].

The black holes constructed here have mass and angular momentum. It is now natural to compute their entropy. In our case, the quickest way to arrive at the right result is by computing the entropy as a Euclidean Noether charge at the horizon [12]. Imposing at the horizon $\beta=(2\pi l^2r_+)/(r_+^2-r_-^2)$ and $\Omega=r_-/(lr_+)$, ensuring the absence of conical singularities, one finds

$$S=Q[e^a_t(r_+),w^{ab}_t(r_+)]=4\pi kr_-. \tag{21}$$

This result is rather surprising because it does not give an entropy proportional to the area of S_1 $(2\pi r_+)$. This is not a contradiction. In fact, for Lagrangians with higher order terms in the curvature the entropy is not proportional to the area [13]. In this case, besides the higher order curvature terms, the relevant term in the action has a non-standard parity with respect to the geometric variables. (In the model discussed in [11] the same value for the entropy was found.) The entropy given in Eq. (21) satisfies the first law

$$\delta M=T\,\delta S+\Omega\,\delta J, \tag{22}$$

where M and J are given in Eqs. (19), (20) and $T=1/\beta$.

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