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Bubbling solutions for Moser-Trudinger type equations on compact Riemann surfaces

Pablo Figueroa and Monica Musso

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Abstract

We study an elliptic equation related to the Moser-Trudinger inequality on a compact Riemann surface \((S, g)\),

\[
\Delta_g u + \lambda \left( u e^{u^2} - \frac{1}{|S|} \int_S u e^{u^2} dv_g \right) = 0, \quad \text{in} \ S, \quad \int_S u dv_g = 0,
\]

where \(\lambda > 0\) is a small parameter, \(|S|\) is the area of \(S\), \(\Delta_g\) is the Laplace-Beltrami operator and \(dv_g\) is the area element. Given any integer \(k \geq 1\), under general conditions on \(S\) we find a bubbling solution \(u_\lambda\) which blows up at exactly \(k\) points in \(S\), as \(\lambda \to 0\). When \(S\) is a flat two-torus in rectangular form, we find that either seven or nine families of such solutions do exist for \(k = 2\). In particular, in any square flat two-torus actually nine families of bubbling solutions with two bubbling points do exist. If \(S\) is a Riemann surface with non-constant Robin’s function then at least two bubbling solutions with \(k = 1\) exists.

Keywords: Moser-Trudinger inequality. Green’s function.

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1 Introduction

Let \((S, g)\) be a compact, orientable Riemann surface. We denote by \(|S|\) the area of \(S\), \(\Delta_g\) the Laplace-Beltrami operator on \(S\) and \(dv_g\) the area element. This paper is devoted to the construction of solutions to the problem

\[
\begin{cases}
\Delta_g u + \lambda \left( u e^{u^2} - \frac{1}{|S|} \int_S u e^{u^2} dv_g \right) = 0, \quad \text{in} \ S, \\
\int_S u dv_g = 0,
\end{cases}
\]

for any values of the small parameter \(\lambda > 0\). These solutions turn out to blow-up, as the parameter \(\lambda \to 0^+\) at very specific points of \(S\).

Problem (1.1) is related to the Trudinger-Moser inequality \([29]\) over a compact Riemann surface \((S, g)\), which can be stated as follows

\[
\sup \left\{ \int_S e^{4\pi u^2} dv_g : u \in H^1(S), \int_S u dv_g = 0 \text{ and } \int_S |\nabla u|^2 dv_g = 1 \right\} < +\infty.
\]
This type of inequality was first proved in [22] on compact Riemannian manifolds of any dimension \( n \). When the dimension is two, this inequality was proved in [24] on manifolds with and without boundary, and the existence of extremal functions was established. We refer also to [25, 26, 34, 35] for related results and generalizations.

It is simple to see that critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (1.1). Our purpose then is to study the existence of solutions to (1.1) for \( \lambda \) positive and small and to describe their asymptotic behavior as \( \lambda \to 0^+ \).

Weak solutions of (1.1) are critical points of the following energy functional

\[
J_\lambda(u) = \frac{1}{2} \int_S |\nabla u|^2 \, dv_g - \frac{\lambda}{2} \int_S e^{u^2} \, dv_g, \quad u \in \tilde{H},
\]

where \( \tilde{H} = \{ u \in H^1(S) : \int_S u \, dv_g = 0 \} \), which corresponds to the free energy associated to the critical Trudinger embedding in the sense of Orlicz spaces [30, 32, 33].

\[
\tilde{H} \ni u \mapsto e^{u^2} \in L^p(S) \quad \forall p \geq 1.
\]

The energy functional (1.2) is thus well defined and it has a Mountain Pass geometric structure. Nevertheless, it is characterized by lack of compactness, which makes it impossible to search for critical points of (1.2) using the classical tools of the Calculus of Variations or of the Critical Point Theory. Indeed, loss of compactness translates into the presence of non-convergent Palais-Smale sequences for the corresponding functional and space of functions.

To better understand this, let us consider the flat case, namely, when \( S = \Omega \subset \mathbb{R}^2 \) is a bounded domain. The Trudinger-Moser inequality concerns the limiting case \( p = 2 \) of the Sobolev embeddings \( W^{1,p}(\Omega) \subset L^{2^*_p}(\Omega) \). It states that there exists \( C_2 > 0 \) such that

\[
\sup \left\{ \int_{\Omega} e^{\alpha |v|^2} \, dx \right\} \int_{\Omega} e^{\alpha |v|^2} \, dx \begin{cases} \leq C_2 |\Omega|, & \text{if } \alpha \leq \alpha_2 \\ = +\infty, & \text{if } \alpha > \alpha_2 \end{cases},
\]

where \( |\Omega| \) is the area of \( \Omega \) and \( \alpha_2 = 4\pi \). After a simple scaling, critical points of the above constrained variational problem satisfy the equation

\[
\Delta u + \lambda e^{u^2} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \lambda > 0 \), whose associated energy functional is

\[
I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega e^{u^2}, \quad u \in H^1_0(\Omega).
\]

For the functional \( I_\lambda \) a precise classification of all Palais-Smale sequences does not seem possible after the results in [2]. Some information is available for sequences of solutions to (1.3), thanks to the result in [17], that states

**Assume that \( u_n \) solves problem (1.3) for \( \lambda = \lambda_n \), with \( I_{\lambda_n}(u_n) \) bounded and \( \lambda_n \to 0 \) as \( n \to +\infty \). Then, passing to a subsequence if necessary, there is an integer \( k \geq 0 \) such that as \( n \to +\infty \)

\[
I_{\lambda_n}(u_n) = 2\pi k + o(1).
\]

A more precise characterization of the sequence of solutions \( (u_n)_n \) is known when \( k = 1 \), see [1]: for all large \( n \), the solution \( u_n \) has only one isolated maximum, whose value diverges to \( +\infty \) as \( \lambda_n \to 0 \), which is attained around a very specific point \( x_0 \in \Omega \). In fact, \( x_0 \) is a critical point of Robin’s function, defined as \( x \mapsto H_\Omega(x, x) \), where \( H_\Omega \) is the regular part of the corresponding Green’s function for the homogeneous Dirichlet problem in \( \Omega \).

Concerning existence of solutions to (1.3) satisfying (1.4), in [2], it is proven that there is a \( \lambda_0 > 0 \) such that a solution to (1.3) exists whenever \( 0 < \lambda < \lambda_0 \) (this is in fact true for a larger
class of nonlinearities with critical exponential growth). By construction this solution falls into the bubbling category \((1.4)\) with \(k = 1\) as \(\lambda \to 0\). If \(\Omega\) has a sufficiently small hole, Struwe in [31] built a solution taking advantage of the presence of topology. This solution exists for a class of nonlinearities, perturbation of the Trudinger-Moser one, that also include \(\lambda u e^{u^2-u}\) for which no solution exists for small \(\lambda\), in a disk, see [312]. It is reasonable to believe that the construction of Struwe in reality produces a second solution of equation \((1.3)\), but this is not known yet. Similar results to [4, 17] on compact Riemann surfaces are obtained in [34].

In [15] authors addressed the existence of bubbling solutions for \((1.3)\) as \(\lambda \to 0\) when \(\Omega\) is not contractible to a point, and \(k\) in \((1.4)\) is any integer number. They provide sufficient conditions for the existence of solutions to \((1.3)\) for small \(\lambda\), which satisfy the bubbling condition \((1.4)\) and give a precise characterization of its bubbling location. In particular, they show that if \(\Omega\) has a hole of any size, namely, \(\Omega\) is not simple connected then at least one of such a solutions exists with \(k = 2\) and if \(\Omega\) has \(d \geq 1\) holes, then \(d + 1\) solutions with \(k = 1\) exist.

The question we address in this paper is whether it is possible to construct a family of solutions \(u_\lambda\) to problem \((1.1)\), for any \(\lambda > 0\) small, whose energy \(J_\lambda(u_\lambda),\) defined in \((1.2)\), is quantized in the sense of \((1.4)\), and whose asymptotic behavior resembles a bubbling phenomena at points, for \(\lambda \to 0\).

In order to state our general result, let us introduce some notations. For a given Riemann surface \((S, g)\), we introduce the Green’s function \(G(x, p)\) with pole at \(p \in S\) as the solution of

\[
\begin{cases}
-\Delta_g G(\cdot, p) = \delta_p - \frac{1}{|S|} & \text{in } S \\
\int_S G(x, p) \, dv_g = 0.
\end{cases}
\]  

(1.5)

Let \(k \geq 1\) be an integer, \(\xi_1, \xi_2, \ldots, \xi_k \in S\) be \(k\) distinct points and \(m_1, m_2, \ldots, m_k\) be \(k\) positive numbers. We define the following functional

\[
\psi_k(\xi, m) = (\log 16 - 2) \sum_{j=1}^{k} m_j^2 + \sum_{j=1}^{k} m_j^2 \log m_j^2 - 4\pi \sum_{j=1}^{k} m_j^2 H(\xi_j, \xi_j) - 4\pi \sum_{i=1}^{k} \sum_{j=1, j \neq i}^{k} m_i m_j G(\xi_i, \xi_j),
\]  

(1.6)

where \(\xi = (\xi_1, \ldots, \xi_k)\) and \(m = (m_1, \ldots, m_k)\). Here, \(G\) is the Green’s function for the Laplace-Beltrami operator on \(S\) given by \((1.3)\) and \(H\) is its regular part. Let us consider an open set \(D\) compactly contained in the domain of the functional \(\psi_k\), namely

\[D \subset \{(\xi, m) \in S^k \times R_+^k \mid \xi_i \neq \xi_j \forall i \neq j \} .\]

We say that \(\psi_k\) has a stable critical point situation if there exists a \(\delta > 0\) such that for any \(g \in C^1(D)\) with \(||g||_{C^1(D)} < \delta\), the perturbed functional \(\psi_k + g\) has a critical point in \(D\).

We can now state our general result.

**Theorem 1.1.** Let \((S, g)\) be a compact, orientable Riemann surface. Let \(k \geq 1\) and assume that there is an open set \(D\) where \(\psi_k\) has a stable critical point situation. Then, for all small \(\lambda > 0\) there exists a family of solutions \(u_\lambda\) of problem \((1.1)\) such that as \(\lambda \to 0\)

\[
\frac{1}{2} \int_S |\nabla u_\lambda|^2 \, dv_g - \frac{\lambda}{2} \int_S e^{u_\lambda^2} \, dv_g = 2k\pi + O(\lambda).
\]  

(1.7)
Moreover, there exists \((\xi_1, m_1) \in \mathcal{D}\), with \(\xi_i = (\xi^{(i)}_1, \ldots, \xi^{(i)}_k)\) and \(m_i = (m^{(i)}_1, \ldots, m^{(i)}_k)\) such that, passing to a subsequence, \((\xi_i, m_i) \to (\xi_0, m_0)\) with \(\nabla \psi_k(\xi_0, m_0) = 0\) and

\[
u_{\lambda}(x) = \sqrt{\lambda} \left( 8\pi \sum_{j=1}^{k} m^{(j)}_\lambda G(x, \xi^{(j)}_\lambda) + O(\lambda) \right)
\]

as \(\lambda \to 0\), uniformly on compact subsets of \(S \setminus \{\xi_1, \ldots, \xi_k\}\).

Concrete examples of surfaces \(S\) on which problem \((\mathbf{11})\) has solutions satisfying \((\mathbf{14})-\mathbf{(18)}\) depends on the possibility to ensure the existence of special critical points for the function \(\psi_k\) defined in \((\mathbf{10})\).

To start with, we observe that if \(S\) is a compact Riemann surface with non-constant Robin’s function then at least two bubbling solutions with \(k = 1\) exists. Indeed, since \(S\) is compact then \(H(\xi, \xi)\) attains its minimum and its maximum. In this case, it is easy to show that \(\psi_1\) has two stable critical point situations. Thus, problem \((\mathbf{11})\) has one solution which is bubbling near the global minimizer of \(H(\xi, \xi)\), and another solution which is bubbling near the global maximizer, as \(\lambda \to 0^+\). Unfortunately, this kind of solutions are hopeless to be found for instance when \(S\) is the unit sphere \(S^2\) in \(\mathbb{R}^3\) or when \(S\) is the flat two-torus \(T\), since, in these examples, the function \(H(\xi, \xi)\) is constant. Nevertheless, in these two examples, we can prove the existence of solutions with \(k = 2\) bubbling points. The case of the flat torus is particularly surprising.

If \(k = 2\), the functional \(\psi_k\) in \((\mathbf{10})\) takes the simplified form

\[
\psi_2(\xi, m) = A \sum_{j=1}^{2} m_j^2 + \sum_{j=1}^{k} m_j^2 \log m_j^2 - 8\pi m_1 m_2 G(\xi_1, \xi_2)
\]

where \(A = \log 16 - 2 - 8\pi c\), where \(c\) is the constant value of \(H(\xi, \xi)\) in the case of the sphere and the flat torus.

Let us start with \(S = S^2\). Since problem \((\mathbf{11})\) is invariant under rotations, it is not restrictive to look for solutions with one bubbling point to be a fixed point on \(S^2\), say \(\xi_2\). Indeed, by a rotation, one can get another solution bubbling at any other point of \(S^2\), just rotating \(\xi_2\) up to this other point. Thus, we fix \(\xi_2 \in S^2\) and we are reduced to study the existence of critical points for \(\xi_1 \in S^2 \mapsto G(\xi_1, \xi_2)\). Notice that \(\xi_1 \mapsto G(\xi_1, \xi_2)\) has a global minimum. Then from simple arguments, one sees that \(\psi_2\) has a stable critical point situation. We thus get the validity of

**Theorem 1.2.** Assume that \(S = S^2\) is the unit sphere in \(\mathbb{R}^3\) and fix \(\xi_2 \in S^2\). Then there exists a family of solutions \(u_\lambda\) to problem \((\mathbf{11})\) with two bubbling points such that as \(\lambda \to 0\) the two bubbling points converge to \((\xi_1, \xi_2)\) with \(\xi_1\) the global minimum of \(G(\cdot, \xi_2)\) and

\[
\frac{1}{2} \int_{S^2} |\nabla u_{\lambda}|^2 g_0 \, dv_{g_0} - \frac{\lambda}{2} \int_{S^2} e^{u_{\lambda}^2} \, dv_{g_0} = 4\pi + o(1), \quad \text{as} \quad \lambda \to 0
\]

where \(g_0\) is the standard round metric on \(S^2\).

Let us now discuss the case when \(S = T\) is a rectangle and we look for solutions \(u\) to \((\mathbf{11})\) that are doubly periodic functions on \(\partial T\). The surprising fact of this case is the multiplicity of solutions.

Without loss of generality, assume that, in complex notation,

\[
T = \left\{ z = \frac{a}{2} + \frac{t}{2} i : s, t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}.
\]

(1.9)
with i the imaginary unity. Since the equation is invariant under translations, if u is a solution to (1.1) then u(· + p) is also a solution to (1.1) for any p ∈ T. In this setting, our next result states the existence of families of solutions of the form (1.8), and satisfying (1.7), with i = 2.

The exact number of such solutions can be 7 or 9, depending on the value of

\[ \tau = b/a. \]

Our result states as follows.

**Theorem 1.3.** Assume that T is a rectangle in \( \mathbb{R}^2 \) given by (1.9) and \( \tau = b/a. \) Then there are \( \tau_0 < \tau_1 \) such that if either \( \tau \in (0, \tau_0] \cup [\tau_1, +\infty) \) or \( \tau \in (\tau_0, \tau_1) \), then there is \( \lambda_0 > 0 \) such that for any \( 0 < \lambda \leq \lambda_0 \) there exist either seven or nine different families of doubly periodic on \( \partial T \) bubbling solutions \( u_{\lambda,i} \) respectively to problem (1.1). These solutions satisfy

\[ J_\lambda(u_{\lambda,i}) = \frac{1}{2} \int_T |\nabla u_{\lambda,i}|^2 \, dx - \lambda \int_T e^{\lambda u_{\lambda,i}} \, dx = 4\pi + O(\lambda), \quad (1.10) \]

as \( \lambda \to 0 \). Moreover, there exist bubbling points \( \xi_{\lambda,i} = (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \in T^2 \) and weights \( m_{\lambda,i} = (m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i}) \in \mathbb{R}_+^2 \) such that, passing to a subsequence, \( (\xi_{\lambda,i}, m_{\lambda,i}) \to (\xi_{0,i}, m_{0,i}) \) with \( \nabla \psi_2(\xi_{0,i}, m_{0,i}) = 0 \) and

\[ u_{\lambda,i}(x) = \sqrt{\lambda} \left( 8\pi m^{(1)}_{\lambda,i} G(x, \xi^{(1)}_{\lambda,i}) + 8\pi m^{(2)}_{\lambda,i} G(x, \xi^{(2)}_{\lambda,i}) + O(\lambda) \right), \quad (1.11) \]

as \( \lambda \to 0 \), uniformly on compact subsets of \( T \setminus \{\xi^{(1)}_{0,i}, \xi^{(2)}_{0,i}\} \), where \( \xi_{0,i} = (\xi^{(1)}_{0,i}, \xi^{(2)}_{0,i}) \). Here we intend that \( i \in \{1, \ldots, 7\} \) when \( \tau \in (0, \tau_0] \cup [\tau_1, +\infty) \), and \( i \in \{1, \ldots, 9\} \) when \( \tau \in (\tau_0, \tau_1) \).

The location of the two bubbling points for the solutions in this result is completely determined. Indeed, fixing \( i \in \{1, \ldots, 7\} \) when \( \tau \in (0, \tau_0] \cup [\tau_1, +\infty) \) or \( i \in \{1, \ldots, 9\} \) when \( \tau \in (\tau_0, \tau_1) \), the bubbling points \( \xi^{(1)}_{\lambda,i} \) and \( \xi^{(2)}_{\lambda,i} \) satisfy \( \xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_j \) for some \( j \in \{1, 2, 3\} \) as \( \lambda \to 0 \) where \( p_1, p_2 \) and \( p_3 \) are the half periods of \( T \):

\[ p_1 = \frac{a}{2}, \quad p_2 = \frac{ib}{2}, \quad p_3 = \frac{a + ib}{2}. \quad (1.12) \]

Moreover, we can show some properties of the weights \( m_i \)'s. If \( 0 < \tau \leq \tau_0 \) then

- there are two bubbling points \( (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \) satisfying \( \xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_1 \), and three different pairs of weights \( (m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i}) \) converging to either \( (m_0, m_0), (m_1, m_2) \) or \( (m_2, m_1) \) as \( \lambda \to 0 \) for some \( m_0, m_1 \) and \( m_2 \) with \( m_0 \neq m_1, m_0 \neq m_2 \) and \( m_1 \neq m_2 \) such that they give rise to three bubbling solutions;

- there are two bubbling points \( (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \) satisfying \( \xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_2 \), and only a pair of weights \( (m_{\lambda,i}, m_{\lambda,i}) \) converging to either \( (m_3, m_4) \) as \( \lambda \to 0 \) for some \( m_3 \) such that it gives rise to a bubbling solution; and

- there are two bubbling points \( (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \) satisfying \( \xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_3 \), and three different pairs of weights \( (m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i}) \) converging to either \( (m_4, m_5) \) or \( (m_5, m_4) \) as \( \lambda \to 0 \) for some \( m_4, m_5 \) and \( m_6 \) with \( m_4 \neq m_5, m_4 \neq m_6 \) and \( m_5 \neq m_6 \) such that they give rise to three bubbling solutions.

If \( \tau \geq \tau_1 \) then

- there are two bubbling points \( (\xi^{(1)}_{\lambda,i}, \xi^{(2)}_{\lambda,i}) \) satisfying \( \xi^{(1)}_{\lambda,i} - \xi^{(2)}_{\lambda,i} \to p_1 \), and only a pair of weights \( (m^{(1)}_{\lambda,i}, m^{(2)}_{\lambda,i}) \) converging to either \( (m_0, m_0) \) as \( \lambda \to 0 \) for some \( m_0 \) such that it gives rise to only a bubbling solution;
• there are two bubbling points \((\xi_\lambda^{(1)}, \xi_\lambda^{(2)})\) satisfying \(\xi_\lambda^{(1)} - \xi_\lambda^{(2)} \to p_2\), and three different pairs of weights \((m_{\lambda,1}, m_{\lambda,2})\) converging to either \((m_1, m_1), (m_2, m_3)\) or \((m_3, m_2)\) as \(\lambda \to 0\) for some \(m_1, m_2\) and \(m_3\) with \(m_1 \neq m_2, m_2 \neq m_3\) and \(m_1 \neq m_3\) such that they give rise to three bubbling solutions; and

• there are two bubbling points \((\xi_\lambda^{(1)}, \xi_\lambda^{(2)})\) satisfying \(\xi_\lambda^{(1)} - \xi_\lambda^{(2)} \to p_3\), and three different pairs of weights \((m_{\lambda,1}^{(1)}, m_{\lambda,2}^{(1)})\) converging to either \((m_4, m_4), (m_5, m_6)\) or \((m_6, m_5)\) as \(\lambda \to 0\) for some \(m_4, m_5\) and \(m_6\) with \(m_4 \neq m_5, m_4 \neq m_6\) and \(m_5 \neq m_6\) such that they give rise to three bubbling solutions.

If \(\tau \in (\tau_0, \tau_1)\) then for every \(j = 1, 2, 3\)

• there are two bubbling points \((\xi_{\lambda,j}^{(1)}, \xi_{\lambda,j}^{(2)})\) satisfying \(\xi_{\lambda,j}^{(1)} - \xi_{\lambda,j}^{(2)} \to p_j\), and three different pairs of weights \((m_{\lambda,i,j}^{(1)}, m_{\lambda,i,j}^{(2)})\) converging to either \((m_0,j, m_0,j), (m_1,j, m_2,j)\) or \((m_2,j, m_1,j)\) as \(\lambda \to 0\) for some \(m_0,j, m_1,j\) and \(m_2,j\) such that they give rise to three bubbling solutions.

A very interesting situation in the rectangular case is when \(a = b\) (or \(\tau = 1\)), namely, in case of a square. Recall that \(\tau_0 < 1 < \tau_1\). This fact follows from the analysis in the proof of the previous result, but we highlight it due to the multiplicity of bubbling solutions we obtain: there exist nine different families of doubly periodic on \(\partial T\) bubbling solutions to problem \(1.1\) in any square.

**Theorem 1.4.** Assume that \(T\) is a square in \(\mathbb{R}^2\). Then for any \(\lambda\) small enough and for every half period \(p_j, j \in \{1, 2, 3\}\), see (1.12), there exist bubbling points \(\xi_{\lambda,j} = (\xi_{\lambda,j}^{(1)}, \xi_{\lambda,j}^{(2)}) \in T^2\) and three different pairs of weights \((m_{\lambda,i,j}^{(1)}, m_{\lambda,i,j}^{(2)})\) in \(\mathbb{R}^2_+\), \(i = 1, 2, 3\) giving rise to nine bubbling solutions \(u_{\lambda,i,j}\) satisfying (1.10) as \(\lambda \to 0\) for \(i = 1, 2, 3\) such that, passing to a subsequence, \((\xi_{\lambda,j}, m_{\lambda,i,j}) \to (\xi_{0,j}, m_{0,i,j}), \xi_{\lambda,j}^{(1)} - \xi_{\lambda,j}^{(2)} \to p_j\) and the property (1.11) holds as \(\lambda \to 0\), uniformly on compact subsets of \(T \setminus \{\xi_{0,j}, \xi_{0,j}^{(2)}\}\), where \(\xi_{0,j} = (\xi_{0,j}^{(1)}, \xi_{0,j}^{(2)})\).

Theorems 1.3 and 1.4 follow from the fact that the existence of nondegenerate critical points of \(\psi_2\) is a stable critical point situation. From similar ideas follows Theorem 1.2 studying first critical points of \(\xi_1\)’s and then the weights \(m_i\)’s.

For the case \(k \geq 3\), or the case \(k = 2\) on a surface \(S\) where the function \(H(\xi, \xi)\) is not constant, the analysis of the map \((\xi, m) \to \psi_k(\xi, m)\) is much harder.

We conclude our introduction mentioning the link between the theorems 1.3 - 1.2 and the results contained in [15] on concentration phenomena for the Liouville-type problem

\[
\Delta u + \varepsilon u^p = 0, \text{ in } \Omega, \quad u = 0, \text{ on } \partial \Omega,
\]

where \(\Omega\) is bounded smooth domain in \(\mathbb{R}^2\), see [5, 14, 19] and references therein. Our results are also connected to those for Liouville-type equations on compact Riemann surfaces, see [6, 7, 8, 13, 18, 21]. The fine blow-up structure for Liouville-type equations on domains in \(\mathbb{R}^2\) or on manifolds very close to the bubbling points is similar to that we found in the problems we are discussing in this paper, nevertheless scalings and intermediate regimes are much more subtle for doubly exponential nonlinearities. Even though the choice of our first approximation to our bubbling solutions is inspired by the discovery of the blow-up shapes which was obtained first in [4] and then in [11, 17], in our problem, more accurate information is needed, due to the role of the distinct weights \(m_j\)’s, which were discovered in [15]. In fact, the presence of the weights \(m_j\) marks a strong difference between double exponential nonlinearity and Liouville type nonlinearity.

As in the usual Lyapunov-Schmidt scheme, the strategy of the proof involves linearization about a first approximation, to later reduce the problem to a finite dimensional variational one.
of adjusting the bubbling centers and the corresponding weights. The critical character of this nonlinearity is very much reflected in the delicate error terms left by the first approximation, which makes the linear elliptic theory needed fairly subtle because of the multiple-regime in the error size and adapted to the Riemann surface $S$ through the use isothermal coordinates.

The paper is organized as follows: in Section 2, we construct a first approximation to a solution to (1.1) with the required properties and we estimate the size of the error of approximation with appropriate norms. In Section 3 we describe the scheme of our proofs, by stating the principal results we need, and we give the proof of our Theorems. Section 4 is devoted to the computation of the expansion of the energy functional on the first approximation we constructed in Section 2. Sections A, B and C are devoted to rigorously prove the intermediate results we state in Section 3.

### 2 Approximation of the solution

It is convenient for our purposes to rewrite problem (1.1) by replacing $u = \sqrt{\lambda}v$, so that the problem becomes

$$
\Delta_g v + \lambda \left( ve^{\lambda v^2} - \frac{1}{|S|} \int_S ve^{\lambda v^2} dv_g \right) = 0,
$$

with $\int_S v \, dv_g = 0$. Following [15], to construct approximating solutions of (2.1), the main idea is to use as “basic cells” the functions

$$
u_{\delta, \xi}(x) = w_{\delta}(x - \xi) \quad \delta > 0, \, \xi \in \mathbb{R}^2,$n

where

$$w_{\mu}(y) := \log \frac{8\mu^2}{(\mu^2 + |y|^2)^2}, \quad \mu > 0. \tag{2.2}$$

They are all the solutions of

$$
\begin{cases}
\Delta u + e^u = 0 & \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^u < +\infty,
\end{cases}
$$

and do satisfy the following concentration property:

$$e^{u_{\delta, \xi}} \to 8\pi\delta \xi \quad \text{in measure sense}$$

as $\delta \to 0$. We will use now isothermal coordinates to pull-back $u_{\delta, \xi}$ in $S$ as in [13].

Let us recall that every Riemann surface $(S, g)$ is locally conformally flat, and the local coordinates in which $g$ is conformal to the Euclidean metric are referred to as isothermal coordinates (see for example the simple existence proof provided by Chern [11]). For every $\xi \in S$ it amounts to find a local chart $y_{\xi}$, with $y_{\xi}(\xi) = 0$, from a neighborhood of $\xi$ onto $B_{2r_0}(0)$ (the choice of $r_0$ is independent of $\xi$) in which $g = e^{\tilde{\varphi}_{\xi}}(y_{\xi}(x)) \, dx$, where $\tilde{\varphi}_{\xi} \in C^\infty(B_{2r_0}(0), \mathbb{R})$. In particular, $\tilde{\varphi}_{\xi}$ relates with the Gaussian curvature $K$ of $(S, g)$ through the relation:

$$
\Delta \tilde{\varphi}_{\xi}(y) = -2K(y_{\xi}^{-1}(y))e^{\tilde{\varphi}_{\xi}(y)} \quad \text{for } y \in B_{2r_0}(0). \tag{2.3}
$$

We can also assume that $y_{\xi}, \tilde{\varphi}_{\xi}$ depends smoothly in $\xi$ and that $\tilde{\varphi}_{\xi}(0) = 0, \nabla \tilde{\varphi}_{\xi}(0) = 0$.

We now pull-back $u_{\delta, 0} = w_{\delta}$ in $\xi \in S$, for $\delta > 0$, by simply setting

$$U_{\delta, \xi}(x) = w_{\delta}(y_{\xi}(x)) = \log \frac{8\delta^2}{(\delta^2 + |y_{\xi}(x)|^2)^2}$$

\[7\]
for \( x \in y^{-1}(B_{2r_0}(0)) \). Letting \( \chi \in C_0^\infty(B_{2r_0}(0)) \) be a radial cut-off function so that \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) in \( B_{r_0}(0) \), we introduce the function \( PU_{\delta, \xi} \) as the unique solution of

\[
\begin{cases}
-\Delta_g PU_{\delta, \xi}(x) = \chi_\xi(x)e^{-\varphi_\xi(x)}e^{U_{\delta, \xi}(x)} - \frac{1}{|S|} \int_S \chi_\xi e^{-\varphi_\xi} e^{U_{\delta, \xi}} dv_g & \text{in } S \\
\int_S PU_{\delta, \xi} dv_g = 0,
\end{cases}
\]

(2.4)

where \( \chi_\xi(x) = \chi(y_\xi(x)) \) and \( \varphi_\xi(x) = \varphi_\xi(y_\xi(x)) \). Notice that the R.H.S. in (2.4) has zero average and smoothly depends in \( x \), and then (2.4) is uniquely solvable by a smooth solution \( PU_{\delta, \xi} \).

Let us recall the transformation law for \( \Delta_g \) under conformal changes: if \( \tilde{g} = e^\varphi g \), then

\[
\Delta_{\tilde{g}} = e^{-\varphi} \Delta_g.
\]

(2.5)

Decompose now the Green function \( G(x, \xi), \xi \in S \), as

\[
G(x, \xi) = -\frac{1}{2\pi} \chi_\xi(x) \log |y_\xi(x)| + H(x, \xi),
\]

and by (1.5) then deduce that

\[
\begin{cases}
-\Delta_g H = -\frac{1}{2\pi} \Delta_{\chi_\xi} \log |y_\xi(x)| - \frac{1}{\pi} (\nabla \chi_\xi, \nabla \log |y_\xi(x)|)_g - \frac{1}{|S|} & \text{in } S \\
\int_S H(\cdot, \xi) dv_g = \frac{1}{2\pi} \int_S \chi_\xi \log |y_\xi(\cdot)| dv_g.
\end{cases}
\]

We have used that

\[
\Delta_g \log |y_\xi(x)| = e^{-\varphi_\xi(y)} \Delta \log |y| \bigg|_{y = y_\xi(x)} = 2\pi \delta_\xi
\]

in view of (2.5).

For \( r \leq 2r_0 \) define \( B_r(\xi) = y_\xi^{-1}(B_r(0)) \) and \( A_{r_1, r_2}(\xi) = B_{r_1}(\xi) \setminus B_{r_2}(\xi) \), for \( r_2 < r_1 \leq 2r_0 \). Setting

\[
\Psi_{\delta, \xi}(x) = PU_{\delta, \xi}(x) - \chi_\xi[U_{\delta, \xi} - \log(8\delta^2)] - 8\pi H(x, \xi),
\]

we have the following asymptotic expansion of \( PU_{\delta, \xi} \) as \( \delta \to 0 \), as shown in [18]:

**Lemma 2.1.** The function \( PU_{\delta, \xi} \) satisfies

\[
PU_{\delta, \xi} = \chi_\xi \left[ U_{\delta, \xi} - \log(8\delta^2) \right] + 8\pi H(x, \xi) + O(\delta^2 \log \delta)
\]

(2.6)

uniformly in \( S \). In particular, there holds

\[
PU_{\delta, \xi} = 8\pi G(x, \xi) + O(\delta^2 \log \delta)
\]

locally uniformly in \( S \setminus \{\xi\} \).

The ansatz will be constructed as follows. Given \( k \in \mathbb{N} \), let us consider distinct points \( \xi_j \in S \), \( m_j > 0 \) and \( \delta_j = \mu_j \varepsilon_j > 0 \), \( j = 1, \ldots, k \). In order to have a good approximation, we will assume that the parameters \( \mu_j \)'s and \( \varepsilon_j \)'s are given by

\[
\log(8\mu_j^2) = -2 \log(2m_j^2) + 8\pi H(\xi_j, \xi_j) + 8\pi \sum_{i=1, i \neq j}^k m_im_j^{-1} G(\xi_i, \xi_j), \text{ for all } j = 1, \ldots, k
\]

(2.7)

and

\[
\log \frac{1}{\varepsilon_j} = \frac{1}{2m_j^2} - 2 \log(2m_j^2), \text{ for all } j = 1, \ldots, k.
\]

(2.8)
Up to take $r_0$ smaller, we assume that the points $\xi_j$'s are well separated and $m_j$'s are in a compact subset of $(0, +\infty)$, namely, we choose $\xi = (\xi_1, \ldots, \xi_k) \in \Xi$ and $m = (m_1, \ldots, m_k) \in \mathcal{M}$, where

$$\Xi = \{(\xi_1, \ldots, \xi_k) \in S^k \mid d_g(\xi_i, \xi_j) \geq 4r_0 \ \forall \ i, j = 1, \ldots, k, \ i \neq j\} \quad \text{and} \quad \mathcal{M} = \left[\delta_0, \frac{1}{\delta_0}\right]^k,$$

for some small fixed constant $\delta_0 > 0$. Denote $U_j := U_{\mu_j, \varepsilon_j, \xi_j}$, $j = 1, \ldots, k$. Thus, our approximating solution is

$$V(x) = \sum_{j=1}^{k} m_j P U_j(x), \quad x \in S, \quad (2.9)$$

where $P$ is the projection operator defined by (2.4). Notice that for $\varepsilon_j \rightarrow 0$ there exists a constant $C > 0$ such that the points $\xi = (\xi_1, \ldots, \xi_k) \in \Xi$ and $m = (m_1, \ldots, m_k) \in \mathcal{M}$. We find that in

$$B(\xi) \quad \text{there exists a constant } C > 0 \quad \text{such that} \quad \|h\|_* = \sup_{x \in S} \rho(x)^{-1} |h(x)|,$$

where

$$\rho(x) := \sum_{j=1}^{k} \chi_{B_{r_0}(\xi_j)}(x) \rho_j(x) + 1,$$

with

$$\rho_j(x) = \lambda \left(\frac{r_0|y_{\xi_j}(x)|}{\delta \varepsilon_j |\log \varepsilon_j|^2}\right) \left(1 + |w_j| + w_j^2\right) \varepsilon_j^{-2} e^{w_j(x)}$$

$$\left[1 - \lambda \left(\frac{2\pi_0}{\delta \varepsilon_j |\log \varepsilon_j|^2}\right) \left(1 + \log |y_{\xi_j}(x)|\right) e^{\lambda \varepsilon_j^2 w_j(x)} + \lambda^{-1}\right] \varepsilon_j^{-2} e^{w_j(x)}, \quad (2.12)$$

$\delta > 0$ a large fixed constant and $\chi_A$ is the characteristic function of the set $A$. Thus, we have proven the following fact.

**Lemma 2.2.** Assume (2.7), (2.8). There exists a constant $C > 0$, independent of $\lambda > 0$ small, such that

$$||R||_* \leq C\lambda \quad (2.13)$$

for all $\xi \in \Xi$, and $m \in \mathcal{M}$.

**Proof:** First, notice that for $x \in B_{2r_0}(\xi_j)$

$$V(x) = m_j \left[w_j(x) + \frac{1}{2\lambda m_j^2} + \theta_j(x)\right] \quad (2.14)$$

where

$$\theta_j(x) = 8\pi \left\langle \nabla(H(\cdot, \xi_j) \circ y_{\xi_j}^{-1})(0) + \sum_{i \neq j} m_i m_j^{-1} \nabla(G(\cdot, \xi_i) \circ y_{\xi_i}^{-1})(0), y_{\xi_j}(x)\right\rangle$$

$$+ O(|y_{\xi_j}(x)|^2) + \sum_{i=1}^{k} O(\varepsilon_i^2 |\log \varepsilon_i|). \quad (2.15)$$
Hence, we obtain that in $B_{r_0}(\xi_j)$

$$\lambda V = \frac{1}{2m_j} + \lambda m_j (w_j + \theta_j) \quad \text{and} \quad \lambda V^2 = w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2 + \frac{1}{4 \lambda m_j^2} \quad (2.16)$$

Thus, from (2.16) we have that in $B_{r_0}(\xi_j)$

$$\lambda V e^{\lambda V^2} = \frac{1}{2m_j} e^{1/(4\lambda m_j^2)} [1 + 2\lambda m_j^2 (w_j + \theta_j)] e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2}$$

$$= m_j [1 + 2\lambda m_j^2 (w_j + \theta_j)] e^{-2w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2}, \quad (2.17)$$

in view of $\frac{1}{4\lambda m_j^2} = \log \frac{2m_j^2}{\varepsilon_j^2}$. Furthermore, in $S \setminus \bigcup_{j=1}^k B_{r_0}(\xi_j)$ we have that

$$V(x) = \sum_{j=1}^k m_j [8\pi G(x, \xi_j) + O(\varepsilon_j^2 \log \varepsilon_j)] = O(1), \quad (2.18)$$

so that $\lambda V e^{\lambda V^2} = O(\lambda)$ in $S \setminus \bigcup_{j=1}^k B_{r_0}(\xi_j)$.

On the other hand, from the definition of $V$ it is readily checked that

$$\Delta_g V = -\sum_{j=1}^k m_j \chi_j e^{-\varphi_j} \varepsilon_{j}^{-2} e^{w_j} + \frac{1}{|S|} \sum_{j=1}^k m_j [8\pi + O(\varepsilon_j^2)], \quad (2.19)$$

where $\chi_j = \chi_{\xi_j}, \varphi_j = \varphi_{\xi_j}$ for $j = 1, \ldots, k$ and in view of $e^{U_j} = \varepsilon_j^{-2} e^{w_j}$ and

$$\int_S \chi_j e^{-\varphi_j} \varepsilon_{j}^{-2} e^{w_j} dv_g = \int_{B_{r_0}(0)} \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 + |y|^2)^2} dy + O(\mu_j^2 \varepsilon_j^2) = 8\pi + O(\varepsilon_j^2).$$

Now, let us estimate the integral term. By using (2.18) we find that

$$\lambda \int_{B_{r_0}(\xi_j)} V e^{\lambda V^2} dv_g = \sum_{j=1}^k \int_{B_{r_0}(\xi_j)} \lambda V e^{\lambda V^2} + O(\lambda) \quad (2.20)$$

Now, we write as follows for $\delta > 0$ large enough and fixed (the same as in the definition of $\rho_j$ in (2.12))

$$\lambda \int_{B_{r_0}(\xi_j)} V e^{\lambda V^2} dv_g = \left[ \int_{A_{r_0, \delta \sqrt{\varepsilon_j}}(\xi_j)} + \int_{A_{\delta \sqrt{\varepsilon_j}, \rho_j \log \varepsilon_j}(\xi_j)} + \int_{B_{\rho_j \log \varepsilon_j}(\xi_j)} \right] \lambda V e^{\lambda V^2} dv_g,$$

In $A_{r_0, \delta \sqrt{\varepsilon_j}}(\xi_j)$, we have that uniformly $V(x) = -4m_j \log |y_{\xi_j}(x)| + O(1)$, in view of the expansion in $S \setminus \bigcup_{j=1}^k B_{\sqrt{\varepsilon_j}}(\xi_j)$

$$PU_j(x) = -4\chi_j(x) \log |y_{\xi_j}(x)| + 8\pi H(x, \xi_j) + \chi_j(x) \log \left(1 + \frac{\varepsilon_j^2}{|y_{\xi_j}(x)|^2}\right) + O(\varepsilon_j^2 \log \varepsilon_j),$$

and for $i \neq j$ and $x \in A_{r_0, \delta \sqrt{\varepsilon_j}}(\xi_j)$, $PU_i(x) = 8\pi G(x, \xi_i) + O(\varepsilon_i^2 \log \varepsilon_i) = O(1)$. Hence, we find that

$$\int_{A_{r_0, \delta \sqrt{\varepsilon_j}}(\xi_j)} V e^{\lambda V^2} dv_g = m_j \int_{A_{r_0, \delta \sqrt{\varepsilon_j}}(\xi_j)} [-4 \log |y_{\xi_j}(x)| + O(1)] e^{\lambda m_j^2 |16 \log^2 |y_{\xi_j}(x)| + O(\log |y_{\xi_j}(x)|)|} dv_g$$

$$= m_j \int_{B_{r_0}(0) \setminus B_{\delta \sqrt{\varepsilon_j}(0)}} [-4 \log |y| + O(1)] e^{\lambda m_j^2 |16 \log^2 |y| + O(\log |y|)|} e^{\varphi_j(y)} dy$$

$$= O\left(\int_{B_{r_0}(0) \setminus B_{\delta \sqrt{\varepsilon_j}(0)}} \log |y| e^{16 \lambda m_j^2 \log^2 |y|} dy\right) = O(1),$$
in view of $y = y_\xi(x)$, $e^{2\varepsilon_j} = O(1)$, $\lambda m_j^2 \log |y| = O(1)$ in the considered region and

$$
\int_{B_{\varepsilon_j}(0) \setminus B_{\delta \sqrt{\varepsilon_j}(0)}} |\log |y|| e^{2\lambda m_j^2 \log^2 |y|} dy = 2\pi \int_{\delta \sqrt{\varepsilon_j}}^{r_0} |\log s| e^{2\lambda m_j^2 \log^2 s\,s\,ds} ds
$$

$$
= 2\pi \int_{\log(\delta \sqrt{\varepsilon_j})}^{\log r_0} |t| e^{2\varepsilon_j + 16\lambda m_j^2 t^2} dt \quad (t = \log s)
$$

$$
= O\left( \int_{\log(\delta \sqrt{\varepsilon_j})}^{\log r_0} |t| e^t dt \right) = O(1),
$$

since for $r_0 < 1$ (if necessary), $\frac{1}{2} \log \varepsilon_j + \log \delta \leq t \leq \log r_0 < 0$ and (2.8) implies that

$$(16\lambda m_j^2 t + 2)t \leq (1 + 4\lambda m_j^2 \log(2m_j^2) + 16\lambda m_j^2 \log \delta)t \leq t + \alpha \quad \text{for some constant } \alpha.
$$

Now, we get that $\delta \varepsilon_j |\log \varepsilon_j| \leq |y_\xi(x)| \leq \delta \sqrt{\varepsilon_j}$ implies that

$$
2\lambda \log \varepsilon_j + \lambda \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon_j^2 + \delta^2)^2} \leq \lambda w_j(x) \leq -4\lambda \log |\log \varepsilon_j| + \lambda \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon_j^2 + \delta^2)^2} < 0,
$$

for $\lambda$ small enough, so that, we find that $\lambda w_j = O(1)$ uniformly in $A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)$. Furthermore, it follows that

$$
w_j(1 + \lambda m_j^2 w_j) \leq w_j \left( 1 + 2\lambda m_j^2 \log \varepsilon_j + \lambda \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon_j^2 + \delta^2)^2} \right) \leq \frac{3}{4} w_j + \beta,
$$

for some constant $\beta$ in $A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|$, in view of $2\lambda m_j^2 \log \varepsilon_j = \lambda m_j^2 \log(2m_j^2) - \frac{1}{4}$. Hence, by using (2.17), $\theta_j = O(1)$ and scaling $\varepsilon_j z = y_\xi(x)$, we obtain that

$$
\int_{A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|} \lambda V e^{\lambda V^2} = \left( \int_{A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|} e^{-2\varepsilon_j w_j + \lambda m_j^2 w_j^2} \right) = O \left( \int_{A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|} \varepsilon_j^{-2} e^{\frac{3}{4} w_j} \right)
$$

$$
= O \left( \int_{B_{\delta \sqrt{\varepsilon_j}(0)} \setminus B_{\delta \varepsilon_j}(0)} \exp \left( \frac{3}{4} \log \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2} \right) dz \right)
$$

$$
= O \left( \int_{\delta \sqrt{\varepsilon_j}}^{\delta \varepsilon_j} \left( \frac{8\mu_j^2}{(\mu_j^2 + s^2)^2} \right)^{3/4} s ds \right) = O\left( \int_{\delta \varepsilon_j}^{\delta \sqrt{\varepsilon_j}} \frac{ds}{s^2} \right) = O(\lambda).
$$

In the ball $B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)$, we have that

$$
\lambda \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} V e^{\lambda V^2} = \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} m_j \left[ 1 + 2\lambda m_j^2 (w_j + \theta_j) \right] \varepsilon_j^{-2} e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2}
$$

$$
= m_j \left[ \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} \varepsilon_j^{-2} e^{w_j + \theta_j} \left( 1 + O(\lambda w_j^2 + \lambda |w_j| + \lambda) \right) dv_g 
$$

$$
+ 2\lambda m_j^2 \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} \varepsilon_j^{-2} e^{w_j} O \left( 1 + |w_j| |1 + \lambda w_j^2 + \lambda |w_j|\right) dv_g \right]
$$

$$
= m_j [8\pi + O(\lambda)] \quad (2.21)
$$

in view of $\theta_j = O(1)$, $\lambda(w_j + w_j^2) = O(1)$,

$$
\int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} \varepsilon_j^{-2} e^{w_j + \theta_j} = 8\pi + O\left( \lambda + \sum_{j=1}^{k} \varepsilon_j^2 |\log \varepsilon_j| \right)
$$

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by using (2.19) (and similar expansion for $e^{\theta_j}$), scaling $\varepsilon_j z = y_{\varepsilon_j}(x)$ so that $\varepsilon_j^2 dy = e^{-\varphi_j(x)} dv_g$ and

$$
\int_{B_{\varepsilon_j \log \varepsilon_j}(\xi)} \varepsilon_j^{-2} e^{w_j} (|w_j| + w_j^2 + |w_j|^3) dv_g = O(1),
$$

since $0 \leq |y_{\varepsilon_j}(x)| \leq \delta \varepsilon_j |\log \varepsilon_j|$ implies that

$$
-4 \log |\log \varepsilon_j| + \log \frac{8 \mu_j^2}{\mu_j^2 \log \varepsilon_j^{-2} + \delta^2} \leq w_j(x) \leq \log \frac{8}{\mu_j^2},
$$

namely, $w_j = O(|\log \lambda|)$ in $B_{\delta \varepsilon_j |\log \varepsilon_j|}(\xi_j)$. Therefore, we conclude that

$$
\lambda \int_{S} V e^{\lambda y^2} \, dv_g = 8 \pi \sum_{j=1}^{k} m_j + O(\lambda).
$$

(2.22)

Notice that from (2.8), we find that $\varepsilon_j^2 |\log \varepsilon_j| = O(\lambda)$, for all $j = 1, \ldots, k$.

From (2.19), (2.17), (2.18) and (2.21) it follows that

- in $S \setminus \bigcup_{j=1}^{m} B_{r_0}(\xi_j)$ there holds $R = O(\lambda)$;
- in $B_{r_0}(\xi_j)$, $j \in \{1, \ldots, k\}$, there holds

$$
R = m_j \varepsilon_j^{-2} e^{w_j} \left( [1 + 2 \lambda \mu_j^2 (w_j + \theta_j)] e^{\lambda \mu_j^2 (w_j + \theta_j)^2} + e^{-\varphi_j} \right) + O(\lambda).
$$

Observe that for $x \in B_{r_0}(\xi_j)$ we have that $R = O(\lambda \varepsilon_j^{-2} e^{w_j} + \lambda)$, since $w_j = O(1)$ uniformly in $B_{\varepsilon_j r_0}(\xi_j)$. Moreover, $w_j = O(|\log \lambda|)$ in $B_{\delta \varepsilon_j |\log \varepsilon_j|^2}(\xi_j)$ and hence,

$$
R = \varepsilon_j^{-2} e^{w_j} O\left( \lambda |w_j| + \lambda w_j^2 + \sum_{j=1}^{k} \varepsilon_j |\log \varepsilon_j|^2 \right) + O(\lambda)
$$

in view of $\lambda (w_j + \theta_j)^2 = O(\lambda |\log \lambda|^2)$ and $\theta_j(x) = O(|y_{\varepsilon_j}(x)| + \sum_{j=1}^{k} \varepsilon_j |\log \varepsilon_j|^2)$. Furthermore, from the choice of $\varepsilon_j$, $j = 1, \ldots, k$, (2.8) it follows that

$$
R = m_j \varepsilon_j^{-2} e^{w_j} \left( [ -4 \log (\mu_j^2 \varepsilon_j^2 + |y_{\varepsilon_j}(x)|^2) + 4 \log (2 \mu_j^2) + 2 \log (8 \mu_j^2) + 2 \theta_j ] + \lambda m_j^2 e^{\lambda m_j^2 \varepsilon_j^2 + \lambda \mu_j^2 \theta_j^2 + \theta_j (1 + 2 \lambda m_j^2 w_j) - e^{-\varphi_j}} \right) + O(\lambda)
$$

$$
= \lambda O \left( [ |\log |y_{\varepsilon_j}(x)|| + 1] \varepsilon_j^{-2} e^{w_j + \lambda m_j^2 \varepsilon_j^2} + \lambda^{-1} \varepsilon_j^{-2} e^{w_j + \lambda m_j^2 \varepsilon_j^2} \right) + O(\lambda),
$$

in $A_{r_0, \delta \varepsilon_j |\log \varepsilon_j|^2}(\xi_j)$, in view of $\log (\mu_j^2 \varepsilon_j^2 + |y_{\varepsilon_j}(x)|^2) = 2 \log |y_{\varepsilon_j}(x)| + O(1)$. In particular, $w_j + \frac{1}{2 \lambda m_j^2} = -4 \log |y_{\varepsilon_j}(x)| + O(1)$, so that, in $A_{r_0, \delta \varepsilon_j |\log \varepsilon_j|^2}(\xi_j)$ it holds $\varepsilon_j^{-2} e^{w_j + \lambda \mu_j^2 \varepsilon_j^2} = O(e^{16 \lambda m_j^2 \log^2 |y_{\varepsilon_j}(x)|})$,

in view of $\lambda w_j = O(1)$ uniformly in $B_{2r_0}(\xi_j)$. Hence, the error of approximation satisfies the global bound

$$
|R(x)| \leq C \lambda \rho(x).
$$

This completes the proof. \[\square\]

For simplicity, here and in what follows $f$ designates the nonlinearity

$$
f(v) = \lambda v e^{\lambda v^2}.
$$

(2.23)
Now, we will look for a solution \( v \) of (2.1) in the form \( v = V + \phi \), for some small remainder term \( \phi \). In terms of \( \phi \), the problem (2.1) is equivalent to find \( \phi \in \bar{H} \) so that

\[
\Delta_g \phi + f'(V)\phi - \frac{1}{|S|} \int_{S} f'(V)\phi dv_g = -R - \left[ f(V + \phi) - f(V) - f'(V)\phi \right] - \frac{1}{|S|} \int_{S} [f(V + \phi) - f(V) - f'(V)\phi] dv_g \quad (2.24)
\]

Here, it is clear that \( f'(V) = \lambda e^{\lambda V^2} (1 + 2\lambda V^2) \). However, instead of solving directly the problem (2.24) we shall study a different problem. To this purpose we need to estimate \( f'(V) \). Thus, denoting

\[
K := \sum_{j=1}^{k} \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j}
\]

we have the following result.

**Lemma 2.3.** Assume (2.7) and (2.8). There exists a constant \( C > 0 \), independent of \( \lambda > 0 \) small, such that

\[
\|f'(V) - K\|_* \leq C\lambda \quad (2.26)
\]

for all \( \xi \in \Xi \), and \( m \in \mathcal{M} \).

**Proof:** From (2.16) it follows that in \( B_{\varepsilon_0}(\xi_j) \)

\[
e^\lambda V^2 = 2m_2^2 \varepsilon_j^{-2} e^{w_j+\theta_j+\lambda m_2^2(w_j+\theta_j)^2} \quad \text{and} \quad 2\lambda V^2 = 2(w_j + \theta_j) + 2\lambda m_2^2(w_j+\theta_j)^2 + \frac{1}{2\lambda m_2^2},
\]

so that, in \( B_{\varepsilon_0}(\xi_j) \)

\[
f'(V) = (1 + 2\lambda m_2^2 + 4\lambda m_2^2(w_j+\theta_j) + 4\lambda^2 m_2^4(w_j+\theta_j)^2) \varepsilon_j^{-2} e^{w_j+\theta_j+\lambda m_2^2(w_j+\theta_j)^2}.
\]

Thus, it is clear that \( f'(V) = \varepsilon_j^{-2} e^{w_j}(1 + O(\lambda)) \) uniformly in \( B_{\varepsilon_0}(\xi_j) \) and \( f'(V) = O(\lambda) \) in \( S \setminus \bigcup_{j=1}^{k} B_{\varepsilon_0}(\xi_j) \). Furthermore, we have that

\[
f'(V) - K = \varepsilon_j^{-2} e^{w_j} \left[ (1 + 2\lambda m_2^2 + 4\lambda m_2^2(w_j+\theta_j) + 4\lambda^2 m_2^4(w_j+\theta_j)^2) \varepsilon_j^{-2} e^{w_j+\theta_j+\lambda m_2^2(w_j+\theta_j)^2} - e^{-\varphi_j} \right]
\]

uniformly in \( B_{\varepsilon_0}(\xi_j) \) and \( f'(V) - K = O(\lambda) \) in \( S \setminus \bigcup_{j=1}^{k} B_{\varepsilon_0}(\xi_j) \). Similar to the estimate (2.13), we conclude (2.20).

In order to simplify the arguments, in view of (2.26), we write (2.24) in the form

\[
L(\phi) = -[R + N(\phi)] \quad \text{in} \quad S, \quad (2.27)
\]

where the linear operator \( L \) is defined as

\[
L(\phi) = \Delta_g \phi + K\phi - \frac{1}{|S|} \int_{S} K\phi dv_g, \quad (2.28)
\]

and the nonlinear part \( N \) is given by

\[
N(\phi) = f(V + \phi) - f(V) - f'(V)\phi - \frac{1}{|S|} \int_{S} [f(V + \phi) - f(V) - f'(V)\phi] dv_g + [f'(V) - K]\phi - \frac{1}{|S|} \int_{S} [f'(V) - K]\phi dv_g. \quad (2.29)
\]

Notice that for all \( \phi \in \bar{H} \)

\[
\int_{S} L(\phi) dv_g = \int_{S} N(\phi) dv_g = \int_{S} Rdv_g = 0.
\]
3 Variational reduction and proof of main results

In the so-called nonlinear Lyapunov-Schmidt reduction, an important step is the solvability theory for the linear operator, obtained as the linearization of (2.1) at the approximating solution \( V \), namely, (3.1). In our approach, in order to simplify the arguments we will study the operator \( L \) given in (2.28) under suitable orthogonality conditions. Let us observe that \( L(\phi) = \hat{L}(\phi) + c(\phi) \), for functions \( \phi \) defined on \( S \), with

\[
\hat{L}(\phi) = \Delta_g \phi + K \phi
\]

and \( c(\phi) := -\frac{1}{|S|} \int_S K \phi \, dv_g \), where \( K \) is given by (2.25). Observe that, as \( \lambda \to 0 \), formally the operator \( \hat{L} \), scaled and centered at 0 by setting \( y = y_{\xi_i}(x)/\varepsilon_j \) for \( x \in B_{\varepsilon_j r_0}(\xi_j) \), approaches \( \hat{L}_j \) defined in \( \mathbb{R}^2 \) as

\[
\hat{L}_j(\phi) = \Delta \phi + \frac{8\mu_j^2}{(\mu_j^2 + |y|^2)^2} \phi.
\]

Due to the intrinsic invariances, the kernel of \( \hat{L}_j \) in \( L^\infty(\mathbb{R}^2) \) is non-empty and is spanned by \( Y_{ij} \), \( i = 0, 1, 2 \), where

\[
Y_{ij}(y) = \frac{4\mu_j y_{\xi_i}}{\mu_j^2 + |y|^2}, \quad i = 1, 2, \quad \text{and} \quad Y_{0j}(y) = 2 \frac{\mu_j^2 - |y|^2}{\mu_j^2 + |y|^2}.
\]

Since [14, 19, 15] it is by now rather standard to show the invertibility of \( L \) in a suitable “orthogonal” space, and a sketched proof of it will be given in Appendix A. See also [18] for an extension to a Riemann surface. Furthermore, an important goal in the study of the \( \hat{L} \) operator is to get rid of the presence of the term \( c(\phi) \).

To be more precise, for \( i = 0, 1, 2 \) and \( j = 1, \ldots, k \) introduce the functions

\[
Z_{ij}(x) = Y_{ij}(\frac{y_{\xi_i}(x)}{\varepsilon_j}) = \begin{cases} 
2 \frac{\mu_j^2 \varepsilon_j^2 - |y_{\xi_i}(x)|^2}{\mu_j^2 \varepsilon_j^2 + |y_{\xi_i}(x)|^2} & \text{for } i = 0 \\
4\mu_j \varepsilon_j (y_{\xi_i}(x))_{\xi_j} & \text{for } i = 1, 2.
\end{cases}
\]

and let \( PZ_{ij} \) be the projections of \( Z_{ij} \) as the solutions in \( H \) of

\[
\left\{ \begin{array}{l}
\Delta_g PZ_{ij} = \chi_j \Delta_g Z_{ij} - \frac{1}{|S|} \int_S \chi_j \Delta_g Z_{ij} \, dv_g \quad \text{in } S \\
\int_S PZ_{ij} \, dv_g = 0,
\end{array} \right.
\]

Notice that \(-\Delta_g Z_{ij} = e^{-\varphi_j} \varepsilon_j^{-2} e^{\mu_j} Z_{ij} \) in \( B_{2r_0}(\xi_j) \) for all \( i = 0, 1, 2 \) and \( j = 1, \ldots, k \). In Appendix A we will prove the following result:

**Proposition 3.1.** There exists \( \lambda_0 > 0 \) so that for any points \( \xi = (\xi_1, \ldots, \xi_k) \in \Xi \) and \( m = (m_1, \ldots, m_k) \in \mathcal{M} \), there is a unique solution \( \phi \in H(S) \cap W^{2,2}(S) \) and coefficients \( c_{ij} \in \mathbb{R} \) of

\[
\left\{ \begin{array}{l}
L(\phi) = h + \sum_{i=0}^{2} \left( \sum_{j=1}^{m} c_{ij} \Delta_g PZ_{ij} \right) \quad \text{in } S \\
\int_S \phi \Delta_g PZ_{ij} \, dv_g = 0, \quad \forall i = 0, 1, 2, \, j = 1, \ldots, m
\end{array} \right.
\]
for all $0 < \lambda < \lambda_0$, $h \in C(S)$ with $\|h\|_* < +\infty$ and $\int_S h \, dv_g = 0$. Moreover, the map $(\xi, m) \mapsto (\phi, c_{ij})$ is differentiable in $(\xi, m)$ with

$$
\|\phi\|_\infty \leq C\|h\|_*, \quad \sum_{i=0}^k \sum_{j=1}^k |c_{ij}| \leq C\|h\|_*, \quad (3.4)
$$

$$
\sum_{j=1}^k \left( \sum_{i=1}^2 \varepsilon_j \|\partial_{(\xi_j, l)}\phi\|_\infty + \frac{1}{|\log \varepsilon_j|} \|\partial_m \phi\|_\infty \right) \leq C\|h\|_* \quad (3.5)
$$

for some $C > 0$.

Let us stress that the right hand side of the equation (3.8) of $L(\phi)$ integrates zero.

Let us recall that $v = V + \phi$ solves (2.1) if $\phi \in H$ does satisfy (2.27). Since the operator $L$ is not fully invertible, in view of Proposition 3.1 one can solve the nonlinear problem (2.27) just up to a linear combination of $\Delta_g PZ_{ij}$’s, as explained in the following (see Appendix B for a proof):

**Proposition 3.2.** Let $\delta_0, r_0 > 0$ small and fixed. Then there exist $\lambda_0 > 0$, $C > 0$ such that for $0 < \lambda < \lambda_0$, for any $\xi = (\xi_1, \ldots, \xi_k) \in \Xi$ and $m = (m_1, \ldots, m_k) \in \mathcal{M}$, problem

$$
\left\{ \begin{array}{l}
L(\phi) = -[R + N(\phi)] + \sum_{i=0}^2 \sum_{j=1}^k c_{ij} \Delta_g PZ_{ij} \quad \text{in } S \\
\int_S \phi \Delta_g PZ_{ij} dv_g = 0 \quad \text{for all } i = 0, 1, 2, j = 1, \ldots, k
\end{array} \right. \quad (3.6)
$$

admits a unique solution $\phi(\xi, m) \in \tilde{H} \cap W^{2,2}(S)$ and $c_{ij} = c_{ij}(\xi, m)$, $i = 0, 1, 2$, $j = 1, \ldots, k$ such that

$$
\|\phi\|_\infty \leq C\lambda, \quad \sum_{i=0}^2 \sum_{j=1}^k |c_{ij}| \leq C\lambda \quad (3.7)
$$

where $R$, $N$ are given by (2.10) and (2.29), respectively. Furthermore, the map $(\xi, m) \mapsto \phi(\xi, m) \in C(S)$ is $C^1$ and for $l = 1, \ldots, k$ we have

$$
\|\partial_{\xi_l} \phi\|_\infty \leq \frac{C\lambda}{\varepsilon_l} \quad \text{and} \quad \|\partial_m \phi\|_\infty \leq C. \quad (3.8)
$$

The function $\phi(\xi, m)$ obtained in Proposition 3.2 will be a true solution of (2.27) if $\xi$ and $m$ are such that $c_{ij}(\xi, m) = 0$ for all $i = 0, 1, 2$, and $j = 1, \ldots, k$. This problem is equivalent to finding critical points of the reduced energy

$$
\mathcal{F}_\lambda(\xi, m) = J_\lambda(U(\xi, m) + \tilde{\phi}(\xi, m)), \quad (3.9)
$$

where $J_\lambda$ is given by (1.2), $U(\xi, m) = \sqrt{\lambda} V(\xi, m)$ and $\tilde{\phi}(\xi, m) = \sqrt{\lambda} \phi(\xi, m)$, as stated in (See Appendix C)

**Lemma 3.3.** There exists $\lambda_0$ such that, if $(\xi, m) \in \Xi \times \mathcal{M}$ is a critical point of $\mathcal{F}_\lambda$ with $0 < \lambda < \lambda_0$, then $v = V(\xi, m) + \phi(\xi, m)$ is a solution of (2.1), i.e., $c_{ij}(\xi, m) = 0$ for all $i = 0, 1, 2$, and $j = 1, \ldots, k$.

Once equation (2.1) has been reduced to the search of c.p.’s for $\mathcal{F}_\lambda$, it becomes crucial to show that the main asymptotic term of $\mathcal{F}_\lambda$ is given by $J_\lambda(U)$, for which we also have an expansion. More precisely, in section 5 we will prove
Proposition 3.4. Assume $\mathbb{M} = \mathbb{R}^3$. The following expansion does hold
\[
\mathcal{F}_\lambda(\xi, m) = 2\pi k - \frac{\lambda |S|}{2} + 8\pi \lambda \psi_k(\xi, m) + \theta_\lambda(\xi, m)
\]
in $C^1(\Xi \times \mathcal{M})$ as $\lambda \to 0^+$, where $\psi_k(\xi, m) = \psi_k(\xi_1, \ldots, \xi_k, m_1, \ldots, m_k)$ is given by (1.6) and the term $\theta_\lambda(\xi, m)$ satisfies
\[
|\theta_\lambda(\xi, m)| + \sum_{l=1}^k \left[ \sum_{i=1}^2 |\partial_{(\xi_i)}, \theta_\lambda(\xi, m)| + |\partial_m, \theta_\lambda(\xi, m)| \right] = O(\lambda^2 \log \lambda)
\]
uniformly for points $(\xi, m) \in \Xi \times \mathcal{M}$.

We are now in position to prove the main results stated in the Introduction.

Proof (of Theorem 1.1): According to Lemma 3.3 we have a solution of problem (1.1) if we adjust $(\xi, m)$ so that it is a critical point of $\mathcal{F}_\lambda$ defined by (3.9). This is equivalent to finding a critical point of
\[
\tilde{\mathcal{F}}_\lambda(\xi, m) = \frac{1}{8\pi \lambda} \left[ \mathcal{F}_\lambda(\xi, m) - 2\pi k + \frac{\lambda |S|}{2} \right].
\]
Thanks to Proposition 3.4 we have that the function $\tilde{\mathcal{F}}_\lambda(\xi, m)$ is $C^1$-close to $\psi_k(\xi, m)$ in $\Xi \times \mathcal{M}$ when $\lambda$ is small enough. Now, let $\mathcal{D}$ be the open set such that
\[
\mathcal{D} \subset \{ (\xi, m) \in S^k \times \mathbb{R}_+^k \colon \xi_i \neq \xi_j, \forall i \neq j \},
\]
where $\psi_k$ has a stable critical point situation. Then any $C^1$-perturbation of $\psi_k$ has a critical point in $\mathcal{D}$. Thus, choosing $\lambda_0$ and $\delta_0$ smaller if necessary so that $\mathcal{D} \subset \Xi \times \mathcal{M}$, we conclude that $\tilde{\mathcal{F}}_\lambda$ has a critical point $(\xi_\lambda, m_\lambda)$ in $\mathcal{D}$, for all such small $\lambda$. Therefore
\[
u_\lambda(x) = U(\xi_\lambda, m_\lambda)(x) + \phi(\xi_\lambda, m_\lambda)(x) = \sqrt{\lambda} |V(\xi_\lambda, m_\lambda)(x) + \phi(\xi_\lambda, m_\lambda)(x)|
\]
is a solution to our problem (1.1). The qualitative properties of this solution predicted by Theorem 1.1 are direct consequence of our construction. This concludes the proof.

Proof (of Theorem 1.3): We shall apply the result of Theorem 1.1 for the case $k = 2$ with $S = T$ a flat two-torus in rectangular form given by (1.9). In this case, it holds that the function $H(\xi, \xi)$ is constant. Notice that on $T$ we have invariance under translations, in other words, if $u$ is a solution to (1.1) then $u(\cdot + p)$ is also a solution to (1.1) for any $p \in T$. Furthermore, by the same property, it is know that the Green’s function satisfies $G(\xi_1, \xi_2) = G(\xi_1 - \xi_2, 0)$. Hence, with a slightly abuse of notation we denote $G(z)$ by the Green’s function $G(\cdot, 0)$ and we make the change of variables $z = \xi_1 - \xi_2$. Thus, we are reduced to look for critical points of the functional
\[
\mathcal{F}_\lambda(z, m) = 4\pi - \frac{\lambda |T|}{2} + 8\pi \lambda \psi_2(z, m) + \theta_\lambda(z, m),
\]
with
\[
\psi_2(z, m) = A \sum_{j=1}^2 m_j^2 + 2 \sum_{j=1}^2 m_j^2 \log m_j - 8\pi m_1 m_2 G(z).
\]
where $A$ is an absolute constant $A = \log 16 - 2 - 8\pi H(\xi, \xi)$, and, it is sufficient first to find nondegenerate critical points $z$ of $\psi_2$ (for any $m = (m_1, m_2)$) and hence, to look for nondegenerate critical points $m$ of $\psi_2(z, \cdot)$ (for the latter $z$), so that $(z, m)$ are nondegenerate critical points of $\psi_2$, since this is a stable critical point situation. Therefore, there exist $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ there exist critical points $(\xi_\lambda, m_\lambda)$ of $\mathcal{F}_\lambda$. Let us stress that we can find critical points $z$ of $\psi_2$ independent of $m$, in view of (3.11).
Claim 3.5. If $T$ is a rectangle then there exist exactly three nondegenerate critical points of $\psi_2(\cdot, m)$ for any $m = (m_1, m_2)$. They are the half periods of $T$: $p_1 = \frac{a}{2}, p_2 = \frac{ib}{2}$ (saddle points) and $p_3 = \frac{a + ib}{2}$ (minimum point), with $i$ the imaginary unity.

Proof: It is known that the Green’s function $G$ has exactly three nondegenerate critical points $p_1, p_2$ and $p_3$, which are the half periods of $T$ given by (1.12), see [8]. Hence, choosing $z = p_i$ for some $i = 1, 2, 3$ we have nondegenerate critical points of $\psi_2(\cdot, m)$ in points $z$ for any $m$. □

Now, let us look for critical points in $m = (m_1, m_2)$. Thus, we get

$$\partial_{m_i} \psi_2(z, m) = 2(A + 1)m_i + 4m_i \log m_i - 8\pi m_j G(z), \quad i, j = 1, 2, j \neq i$$

and we have to find points $m_1, m_2 \in (0, +\infty)$ solutions to the system

$$\begin{cases} (A + 1)m_1 + 2m_1 \log m_1 = 4\pi m_2 G(z) \\ (A + 1)m_2 + 2m_2 \log m_2 = 4\pi m_1 G(z) \end{cases}. \tag{3.12}$$

Let us stress that for each $z = p_i, i = 1, 2, 3$ we look for $m$ a critical point of $\psi_2(p_i, \cdot)$.

Claim 3.6. If $G(z) \geq 0$ then there exists an only solution $(m_1, m_2)$ to the system (3.12) and it satisfies $m_1 = m_2$. If $G(z) < 0$ then there exist exactly three different pairs of solutions $(m_1, m_2)$ to the system (3.12) in the form $(m_1, m_2)$ or $(m_0, m_0)$ or $(m_2, m_1)$ with some positive real numbers satisfying $m_1 < m_0 < m_2$.

Proof: To this aim, denote $B = 4\pi G(z)$ and first assume that $B \neq 0$. Consider the function

$$f_0(t) = \frac{A + 1}{B} t + 2 \frac{t}{B} \log t \tag{3.13}$$

so that we re-write (3.12) in the form

$$\begin{cases} f_0(t) = s \\ f_0(s) = t \end{cases}. \tag{3.14}$$

Thus, we look for the intersection points between the two curves $s = f_0(t)$ and $t = f_0(s)$ in the plane $ts$. Note that $f_0$ satisfies $f_0(0) = 0$, its derivative is $f_0'(t) = \frac{A + 1}{B} + 2 \frac{t}{B} \log t + \frac{2}{B}$ and hence, $f_0'$ is strictly increasing if $B > 0$ and strictly decreasing if $B < 0$, so that $f_0$ is strictly convex if $B > 0$ and strictly concave in $[0, +\infty)$ if $B > 0$ and

$$f_0'(t) \to \begin{cases} -\infty, & B > 0 \\ +\infty, & B < 0 \end{cases} \text{ as } t \to 0^+ \text{ and } f_0'(t) \to \begin{cases} +\infty, & B > 0 \\ -\infty, & B < 0 \end{cases} \text{ as } t \to +\infty.$$

Now, assume that $B > 0$, namely, $G(z) > 0$. From the previous analysis, $f_0$ satisfies that $f(0) = 0$, $f_0$ is strictly convex in $[0, +\infty)$, $f_0'(t) \to -\infty$ as $t \to 0^+$ and $f_0'(t) \to +\infty$ as $t \to +\infty$. Furthermore, it is readily check that $f_0$ is strictly decreasing in $[0, e^{-\frac{(A+3)/2}}]$, its graph is below the axis $t$ in $[0, e^{-\frac{(A+3)/2}}]$ ($f_0$ is negative), it is strictly increasing in $[e^{-\frac{(A+3)/2}}, +\infty)$ and it has a minimum at $t = e^{-\frac{(A+3)/2}}$. Therefore, there is a unique $m_0 = m_0(B) > 0$ such that $f_0(m_0) = m_0$, namely, $m_0$ is the only fixed point of $f_0$, it satisfies

$$(A + 1)m_0 + 2m_0 \log m_0 = Bm_0$$

and $(m_0, m_0)$ is the only solution to system (3.12), since reflecting the curve $s = f_0(t)$ with respect to the line $s = t$ we obtain the curve $t = f_0(s)$ and $(m_0, m_0)$ is the only point of intersection of the curves.
Hence, we deduce that there are three distinct points of intersection between the curves \( m \) that reflecting the curve \( s \). If \( m \) is nondegenerate we study its Hessian matrix. Notice that if \( G \) is a nondegenerate critical point of \( (3.13) \), when \( |H_m\psi_2(z,\cdot)| \geq 3 \), it is strictly concave in \([0, \pi]\), its graph is above the diagonal \( s = t \) and \( t = f_0(s) \). Therefore, there is a unique \( m_0 = m_0(B) > 0 \) such that \( f_0(m_0) = m_0 \) and two more distinct solutions \((m_1, m_2)\) and \((m_2, m_1)\) to the system (3.12) with \( m_1 < m_0 < m_2 \) and satisfying \( f_0(\psi_2(z, f_0(m_1))) = m_1 \) and \( f_0(\psi_2(z, m_2)) = m_2 \), namely, with \( m_1 \) and \( m_2 \) are fixed points of \( f_0 \circ f_0 \) (but \( f_0(m_1) \neq m_1 \) and \( f_0(m_2) \neq m_2 \)). Also, it follows that \( m_1 < e^{-(A+3)/2} < m_0 \). Let us stress that \( m_0 \) is a fixed point of \( f_0 \) and \( m_1 \) and \( m_2 \) are fixed points of \( f_0 \circ f_0 \) but not of \( f_0 \).

Now, for the case \( B = 0 \), system (3.12) is reduced to solve the equation

\[
(A + 1)t + 2t \log t = 0 \iff t(A + 1 + 2 \log t) = 0.
\]

Since we look for solutions \( m_1, m_2 > 0 \), we get that \( m_1 = m_2 = e^{-\frac{A+1}{2}} \) is the only solution to the system (3.12) with \( G(z) = 0 \).

The proof of the claim is finished.

To conclude that a critical point \((m_1, m_2)\) of \( \psi_2(z, \cdot) \), namely, a solution to the system (3.12) is nondegenerate we study its Hessian matrix. Notice that if \( G(z) = -\frac{1}{4\pi} \) then the system (3.12) has three pairs of solutions in the form \((m_1, m_2)\) or \((m_0, m_0)\) or \((m_2, m_1)\) with some positive real numbers \( m_1 < m_0 < m_2 < m_2 \).

**Claim 3.7.** If \( G(z) \neq -\frac{1}{4\pi} \) then all the solutions \((m_1, m_2)\) to the system (3.12) are nondegenerate critical points of \( \psi_2(z, \cdot) \). If \( G(z) = -\frac{1}{4\pi} \) then the pairs of solutions \((m_1, m_2)\) and \((m_2, m_1)\) to the system (3.12) are nondegenerate critical points of \( \psi_2(z, \cdot) \) and \((m_0, m_0)\) is degenerate.

**Proof:** Direct computations lead us to find the determinant of the Hessian matrix as

\[
|H_m\psi_2(z, m)| = (8\pi \frac{m_2}{m_1} G(z) + 4) (8\pi \frac{m_1}{m_2} G(z) + 4) - 64\pi G(z))^2 = 32\pi (\frac{m_2}{m_1} + \frac{m_1}{m_2}) G(z) + 16
\]

by using that \((m_1, m_2)\) satisfies (3.12). Therefore, if \( 2\pi (\frac{m_2}{m_1} + \frac{m_1}{m_2}) G(z) \neq -1 \) then \( m = (m_1, m_2) \) is a nondegenerate critical point of \( \psi_2(z, \cdot) \) (for fixed \( z \)). In particular, if \( G(z) \geq 0 \) then we have an only nondegenerate critical point \( m = (m_0, m_0) \), with \( m_0 \) the only fixed point of \( f_0 \) defined by (3.13), when \( B > 0 \) or \( m_0 = e^{-\frac{A+1}{2}} \) when \( B = 0 \), since \( |H_m\psi_2(z, m)| \geq 16 \). In case \( G(z) < 0 \) we have at most three nondegenerate critical points. More precisely, we have several cases readily checked:

- \( m = (m_0, m_0) \), with \( m_0 \) the only fixed point of \( f_0 \) defined by (3.13), is a nondegenerate critical point of \( \psi_2(z, \cdot) \) only if \( G(z) \neq -\frac{1}{4\pi} \), in view of \( |H_m\psi_2(z, m)| = 64\pi G(z) + 16 \neq 0 \); in other words, if \( G(z) < -\frac{1}{4\pi} \) then \( |H_m\psi_2(z, m)| = 64\pi G(z) + 16 < 0 \) and if \( -\frac{1}{4\pi} < G(z) < 0 \) then \( |H_m\psi_2(z, m)| = 64\pi G(z) + 16 > 0 \);
- if \( G(z) = -\frac{1}{4\pi} \), then on one hand, \((m_0, m_0)\), with \( m_0 \) the only fixed point of \( f_0 \) defined by (3.13), is a degenerate critical point of \( \psi_2(z, \cdot) \), in view of \( |H_m\psi_2(z, m)| = 64\pi G(z) + 16 = 0 \), and on the other hand, both \((m_1, m_2)\) and \((m_2, m_1)\), where \( m_1 \) and \( m_2 \) are the fixed points
of \( f_0 \circ f_0 \) different than \( m_0 \) the fixed point of \( f_0 \), with \( m_1 < m_0 < m_2 \), are nondegenerate critical points of \( \psi_2(z, \cdot) \) in view of \( m_1 \neq m_2 \), so that,

\[
2\pi \left( \frac{m_2}{m_1} + \frac{m_1}{m_2} \right) G(z) = -\frac{1}{2} \left( \frac{m_2}{m_1} + \frac{m_1}{m_2} \right) \neq -1; \quad \text{and}
\]

- either \((m_1, m_2)\) or \((m_2, m_1)\), with \( m_1 \) and \( m_2 \) the fixed points of \( f_0 \circ f_0 \) different than \( m_0 \) the fixed point of \( f_0 \) defined by (3.13), with \( m_1 < m_0 < m_2 \), are nondegenerate critical points of \( \psi_2(z, \cdot) \) if \( G(z) < -\frac{1}{4\pi} \) in view of \( |H_m \psi_2(z, m)| \leq 64\pi G(z) + 16 < 0 \), since \( \frac{m_2}{m_1} + \frac{m_1}{m_2} \geq 2 \).

Notice that it remains to analyze the critical points \((m_1, m_2)\) or \((m_2, m_1)\), with \( m_1 \) and \( m_2 \) the fixed points of \( f_0 \circ f_0 \) different than \( m_0 \) the fixed point of \( f_0 \) defined by (3.13) when \(-\frac{1}{4\pi} < G(z) < 0 \). To this aim, we shall use the equation that must satisfy degenerate critical point of \( \psi_2(z, \cdot) \). Thus, we get that

\[
2\pi \left( \frac{m_2}{m_1} + \frac{m_1}{m_2} \right) G(z) + 1 = 0 \iff 4\pi G(z)m_1^2 + 2m_1m_2 + 4\pi G(z)m_2^2 = 0.
\]

Recall that \( B = 4\pi G(z) \), so that, \(-1 < B < 0 \) when \(-\frac{1}{4\pi} < G(z) < 0 \). So, we re-write the latter equality in the form \( Bt^2 + 2ts + Bs^2 = 0 \) and we consider the system

\[
\begin{align*}
(A+1)t + 2t \log t &= Bs, \\
(A+1)s + 2s \log s &= Bt, \\
Bt^2 + 2ts + Bs^2 &= 0
\end{align*}
\tag{3.15}
\]

Let us show that this system does not have any solution, so that the critical points \((m_1, m_2)\) or \((m_2, m_1)\) are nondegenerate. Assume that \((t, s)\) is a solution of the system (3.15). From the third equation it follows that either \( Bt + (1 + \sqrt{1 - B^2}) s = 0 \) or \( Bt + (1 - \sqrt{1 - B^2}) s = 0 \). First, assume that \( Bt + (1 - \sqrt{1 - B^2}) s = 0 \). Hence, we have that

\[
Bt = -\left( 1 - \sqrt{1 - B^2} \right) s \iff Bs = -\left( 1 + \sqrt{1 - B^2} \right) t.
\]

So, replacing \( Bt \) in the second equation of the system (3.15) and since \( s \neq 0 \), we get that

\[
s = \exp \left( - \left[ A + 2 - \sqrt{1 - B^2} \right]/2 \right) \quad \text{and} \quad t = \frac{-1 + \sqrt{1 - B^2}}{B} \exp \left( - \left[ A + 2 - \sqrt{1 - B^2} \right]/2 \right).
\]

On the other hand, similarly as above replacing \( Bs \) in the first equation of the system (3.15) we get that

\[
t = \exp \left( - \left[ A + 2 + \sqrt{1 - B^2} \right]/2 \right) \quad \text{and} \quad s = \frac{-1 - \sqrt{1 - B^2}}{B} \exp \left( - \left[ A + 2 + \sqrt{1 - B^2} \right]/2 \right).
\]

If \((t, s)\) is a solution of the system (3.15) then necessarily

\[
s = \exp \left( - \left[ A + 2 - \sqrt{1 - B^2} \right]/2 \right) = \frac{-1 - \sqrt{1 - B^2}}{B} \exp \left( - \left[ A + 2 + \sqrt{1 - B^2} \right]/2 \right).
\]

From this equality we obtain that

\[
B \exp \left( \sqrt{1 - B^2} \right) + \sqrt{1 - B^2} + 1 = 0.
\tag{3.16}
\]

Performing the change of variable \( t = \sqrt{1 - B^2} \), we have that \( 0 < t < 1 \), \( B = -\sqrt{1 - t^2} \) and \( t \) satisfies the equation

\[
-\sqrt{1 - t^2}e^t + t + 1 = 0.
\]
It turns out that the function \( g(t) = \sqrt{1 - t^2} e^t \) satisfies \( g(0) = 1, \ g'(0) = 1 \), \( g \) is strictly concave and its tangent line at \( t = 0 \) is exactly \( s = t + 1 \). Therefore, \( g(t) < t + 1 \) for all \( 0 < t < 1 \) and the equality is attained at \( t = 0 \). In other words, there is no \( B \) with \(-1 < B < 0\) satisfying the equation \((3.10)\). In case \( Bt + (1 + \sqrt{1 - B^2})s = 0 \), a similar analysis lead us to find the equation \((3.11)\). Thus, in any case we arrive at a contradiction and the system \((3.12)\) has no solutions. Therefore, if \( G(z) \neq -\frac{1}{4\pi} \) then all the critical points are nondegenerate. This finishes the proof of the claim.

From the previous result it remains to study the case \( G(z) = -\frac{1}{4\pi} \) for some \( z \in \{p_1, p_2, p_3\} \), since we have that there exist three critical points of \( \psi_2 \) but one of them is degenerate. We shall address this difficulty by study directly the functional \( \mathcal{F}_\lambda \) in that case.

**Claim 3.8.** Assume that \( G(z) = -\frac{1}{4\pi} \) for some fixed \( z \in \{p_1, p_2, p_3\} \), then there exist three critical points \((z_{\lambda,i}, m_{\lambda,i})\) \( i = 1, 2, 3 \), of \( \mathcal{F}_\lambda(z, m) \) such that, up to subsequences, as \( \lambda \to 0 \) \( z_{\lambda,1} \to z \) and \( m_{\lambda,1} \to m_0,1 \) with \( m_{0,1} = (m_1, m_2), m_{0,2} = (m_0, m_0) \) and \( m_{0,3} = (m_2, m_1) \) the three different pairs of solutions to the system \((3.12)\) obtained in Claim 3.6.

**Proof:** We know that \( z \) is a nondegenerate critical point of \( \psi_2(\cdot, m) \). This is a stable critical point situation so that for each \( m \in (0, +\infty) \times (0, +\infty) \) there exist a critical point \( z_\lambda(m) \in T \) of \( \tilde{F}_\lambda(\cdot, m) \) with

\[
\tilde{F}_\lambda(z, m) = \frac{1}{8\pi\lambda} \left[ F_\lambda(z, m) - 4\pi + \frac{\lambda |T|}{2} \right],
\]

so that \( z_\lambda(m) \to z \) as \( \lambda \to 0 \). Let us stress that \( \tilde{F}_\lambda(z, m) = \psi_2(z, m) + O(\lambda |\log \lambda|) \), where \( O \) is uniformly in \( C^1 \)-sense for points in \( T \times \Xi \). Moreover, by IFT the map \( m \in \Xi \to z_\lambda(m) \) is a \( C^1 \)-function in \( m \). Now, let us define \( \mathcal{E}_\lambda(m) = \mathcal{F}_\lambda(z_\lambda(m), m) \). Then, it readily follows that

\[
\nabla_m \mathcal{E}_\lambda(m) = \nabla_m \tilde{F}_\lambda(z_\lambda(m), m) = \nabla_m \psi_2(z_\lambda(m), m) + O(\lambda |\log \lambda|),
\]

since \( \nabla \tilde{F}_\lambda(z_\lambda(m), m) = 0 \). Hence, we look for critical points of \( \mathcal{E} \) by study the system \( \nabla_m \mathcal{E}(m) = 0 \). This is equivalent to finding solutions to the perturbation of the system \((3.12)\) given by

\[
\begin{cases}
(A + 1)m_1 + 2m_1 \log m_1 - 4\pi m_2 G(z_\lambda(m_1, m_2)) + \lambda |\log \lambda| \tilde{\theta}_{\lambda,1}(m_1, m_2) = 0, \\
(A + 1)m_2 + 2m_2 \log m_2 - 4\pi m_1 G(z_\lambda(m_1, m_2)) + \lambda |\log \lambda| \tilde{\theta}_{\lambda,2}(m_1, m_2) = 0,
\end{cases}
\]

where it holds \( \tilde{\theta}_{\lambda,i} = O(1), \ i = 1, 2 \) uniformly for \( m \in \Xi \) and \( 4\pi G(z_\lambda(m_1, m_2)) \to -1 \) as \( \lambda \to 0 \). Since system \((3.12)\) with \( 4\pi G(z) = -1 \) has exactly three different pairs of solutions, as proved in Claim 3.6, it follows that for \( \lambda \) small enough there are at least three different pairs \( m_{\lambda,i} \) of solutions to \((3.17)\) such that as \( \lambda \to 0 \) converge, up to a subsequence, to a solution to \((3.12)\). Let us stress that we can consider the curves (or the implicit functions)

\[
4\pi m_j G(z_\lambda(m_1, m_2)) = (A + 1)m_i + 2m_i \log m_i + \lambda |\log \lambda| \tilde{\theta}_{\lambda,i}(m_1, m_2)
\]

for \( i \neq j \) converging uniformly to the curves (or the implicit functions)

\[
-m_j = (A + 1)m_i + 2m_i \log m_i
\]

for \( i \neq j \) locally around each \( m_0, m_1, m_2 \) in order to obtain the existence of the pairs \( m_{\lambda,i} \), \( i = 1, 2, 3 \). Thus, we conclude that \( m_{\lambda,i} \), \( i = 1, 2, 3 \) are critical points of \( \mathcal{E}_\lambda \), namely,

\[
\nabla_m \mathcal{E}(m_{\lambda,i}) = 0 \quad \text{and} \quad m_{\lambda,i} \to m_{0,i} \quad \text{with} \quad m_{0,1} = (m_1, m_2), \ m_{0,2} = (m_0, m_0) \quad \text{and} \quad m_{0,3} = (m_2, m_1)
\]

the three different pairs of solutions to the system \((3.12)\) obtained in Claim 3.6. By the procedure it follows that \((z_\lambda(m_{\lambda,i}), m_{\lambda,i})\), \( i = 1, 2, 3 \) are critical points of \( \tilde{F}_\lambda \), in view of

\[
\nabla_z \tilde{F}_\lambda(z, m_{\lambda,i}), m_{\lambda,i} = 0 \quad \text{and} \quad \nabla_m \tilde{F}_\lambda(z, m_{\lambda,i}), m_{\lambda,i} = \nabla_m \mathcal{E}(m_{\lambda,i}) = 0.
\]

The proof of the Claim is finished. □
In order to complete the study of existence of two bubbling solutions to \( \text{(1.1)} \) in the flat two-torus in rectangular form \( T \), we shall show that the multiplicity depends on the values \( G(z) \) with \( z \in \{ p_1, p_2, p_3 \} \), precisely, depends on the form of \( T \). Recall, \( G \) has three nondegenerate critical points: \( p_1 = \frac{a}{2}, p_2 = \frac{ib}{2} \) (saddle points) and \( p_3 = \frac{a+ib}{2} \) (minimum point). Notice that since \( G \) has zero average and \( p_3 \) is a minimum point we have that \( G(p_3) < 0 \) for any \( a,b > 0 \). In other words, \( G\left(\frac{a+ib}{2}\right) \leq G\left(\frac{a}{2}\right) \) and \( G\left(\frac{a+ib}{2}\right) \leq G\left(\frac{ib}{2}\right) \). From an explicit formula for \( G \) shown in [10], direct computations lead us to get that

\[
G(p_1) = f_i\left(\frac{b}{a}\right), \quad i = 1, 2, 3
\]

where \( f_i, i = 1, 2, 3 \) are given by

\[
f_1(\tau) = \frac{\tau}{12} - \frac{1}{2\pi} \log 2 - \frac{1}{\pi} \sum_{n=1}^{+\infty} \log \left(1 + e^{-2\pi n \tau}\right),
\]

\[
f_2(\tau) = -\frac{\tau}{24} - \frac{1}{\pi} \sum_{n=0}^{+\infty} \log \left(1 - e^{-\pi(2n+1)\tau}\right)
\]

and

\[
f_3(\tau) = -\frac{\tau}{24} - \frac{1}{\pi} \sum_{n=0}^{+\infty} \log \left(1 + e^{-\pi(2n+1)\tau}\right),
\]

so that we can study them in terms of \( \tau = \frac{b}{a} \). By symmetry arguments it follows that in case \( a = b \), namely, \( \tau = 1 \), it holds \( G\left(\frac{a}{2}\right) = G\left(\frac{ia}{2}\right) \), so, equivalently \( f_1(1) = f_2(1) \approx -0.03 \). By studying \( f_i, i = 1, 2, 3 \) we obtain the following fact.

**Claim 3.9.** There exist \( \tau_0 < 1 < \tau_1 \) such that \( f_1(\tau_1) = 0, f_2(\tau_0) = 0 \). If \( \tau \in (0, \tau_0] \cup [\tau_1, +\infty) \) then there exist seven different critical points \( (\xi_{\lambda,1}, m_{\lambda,1}) \) \( i = 1, \ldots, 7 \) of \( \mathcal{F}_\lambda \). If \( \tau \in (\tau_0, \tau_1) \) then there exist nine different critical points \( (\xi_{\lambda,i}, m_{\lambda,i}) \) \( i = 1, \ldots, 9 \) of \( \mathcal{F}_\lambda \).

**Proof:** From the definition of \( f_1 \) it follows that \( f_1 \) is a continuous function, strictly increasing and strictly concave, \( f_1(0) \approx -1.43 \) and \( f_1(\tau) \to +\infty \) as \( \tau \to +\infty \), so that, there is \( \tau_1 > 1 \) such that \( f_1(\tau_1) = 0 \). Also, \( f_2 \) is a continuous function, strictly decreasing and strictly convex, with \( f_2(\tau) \to +\infty \) as \( \tau \to 0^+ \) and \( f_2(\tau) \to -\infty \) as \( \tau \to +\infty \) so that there is \( \tau_0 < 1 \) such that \( f_2(\tau_0) = 0 \). For \( f_3 \) we obtain that it is a continuous function, strictly concave, \( \tau = 1 \) is a maximum with \( f_3(1) \approx -0.06 \), \( f_3(0) \approx -1.32 \) and \( f_3(\tau) \to -\infty \) as \( \tau \to +\infty \).

Hence, depending on the value of \( \tau = \frac{b}{a} \) we have three cases.

1. If \( \tau \in (0, \tau_0] \) then \( f_1(\tau) = G\left(\frac{a}{2}\right) < 0, f_2(\tau) = G\left(\frac{ib}{2}\right) \geq 0 \) and \( f_3(\tau) = G\left(\frac{a+ib}{2}\right) < 0 \), so that we have seven critical points of \( \tilde{\mathcal{F}}_\lambda \). Precisely, \( G\left(\frac{a}{2}\right) < 0, G\left(\frac{ib}{2}\right) \geq 0 \) and \( G\left(\frac{a+ib}{2}\right) < 0 \) gives rise to three, one and three critical points respectively.

2. If \( \tau \in (\tau_0, \tau_1) \) then \( G\left(\frac{a}{2}\right) < 0, G\left(\frac{ib}{2}\right) < 0 \) and \( G\left(\frac{a+ib}{2}\right) < 0 \), so that we have nine critical points of \( \tilde{\mathcal{F}}_\lambda \). Precisely, \( G\left(\frac{a}{2}\right), G\left(\frac{ib}{2}\right), G\left(\frac{a+ib}{2}\right) < 0 \) gives rise to three critical points each one.
3. If \( \tau \in (\tau_1, +\infty) \) then \( G\left(\frac{a}{2}\right) \geq 0, G\left(\frac{ib}{2}\right) < 0 \) and \( G\left(\frac{a + ib}{2}\right) < 0 \), so that we have seven critical points of \( \tilde{F}_\lambda \). Precisely, \( G\left(\frac{a}{2}\right) \geq 0, G\left(\frac{ib}{2}\right), G\left(\frac{a + ib}{2}\right) < 0 \) gives rise to one, three and three critical points respectively.

The proof of the Claim is finished \( \square \)

Therefore, given a rectangle we can obtain exactly either seven or nine different family of solutions. This completes the proof. \( \square \)

**Proof (of Theorem 1.2):** Assume that \( S = \mathbb{S}^2 \). By invariance under rotations it follows that \( H(\xi, \xi) \) is constant. Furthermore, as it was said in the introduction problem (1.1) is invariant under rotations, so we look for solutions with one bubbling point fixed. Thus, with a slightly abuse of notation, we are reduced to look for critical points of the functional

\[
\mathcal{F}_\lambda(\xi_1, m) = 4\pi - \frac{\lambda |T|}{2} + 8\pi\lambda \left[ A \sum_{j=1}^{2} m_j^2 + 2 \sum_{j=1}^{2} m_j^2 \log m_j - 8\pi m_1 m_2 G(\xi_1, \xi_2) \right] + \theta_\lambda(\xi_1, m),
\]

where \( A = \log 16 - 2 - 8\pi H(\xi, \xi) \). In other words, we fix \( \xi_2 \in \mathbb{S}^2 \) and look for critical points on \( \xi_1 \) of \( G(\cdot, \xi_2) \). Since \( G(\cdot, \xi_2) \) has a global minimum, for any \( m \) there exist \( \xi_{1,\lambda}(m) \in \mathbb{S}^2 \) such that \( \mathcal{F}_\lambda \) attains its minimum at \( \xi_1 = \xi_{1,\lambda}(m) \). Hence, from the same procedure used in Claim 3.8 it follows that for \( \lambda \) small enough there exists \( m_\lambda \) such that \( (\xi_{1,\lambda}(m_\lambda), m_\lambda) \) is actually a critical point of \( \mathcal{F}_\lambda(\xi_1, m) \). This finishes the proof. \( \square \)

4 Proof of Proposition 3.4

The purpose of this section is to give a proof of the Proposition 3.4, namely, an asymptotic expansion of the “reduced energy” \( \mathcal{F}_\lambda(\xi, m) = J_\lambda(U(\xi, m) + \bar{\phi}(\xi, m)) \), where \( J_\lambda \) is the energy functional given by (1.2), \( U = \sqrt{\lambda} V \) with \( V \) defined by (2.9) and \( \bar{\phi} = \sqrt{\lambda} \phi \) with \( \phi \) the solution given by Proposition 3.2. The proof will be divided into several steps. To this aim, we recall the following result. See [18] for a proof.

**Lemma 4.1.** Letting \( \bar{f} \in C^2(\gamma(S)) \) (possibly depending in \( \xi \)), \( 0 < \gamma < 1 \). The following expansions do hold as \( \delta \to 0 \):

\[
\int_S \chi x e^{-\nu x} \bar{f}(x) e^{U_{\delta}} dv_q = 8\pi \bar{f}(\xi) - 4\pi \delta^2 \log \delta \Delta_g \bar{f}(\xi) + O(\delta^2),
\]

\[
\int_S \chi x e^{-\nu x} \bar{f}(x) e^{U_{\delta}} \frac{dv_q}{\delta^2 + |y_\xi(x)|^2} = \frac{4\pi}{\delta^2} \bar{f}(\xi) + \pi \Delta_g \bar{f}(\xi) + O(\delta^\gamma)
\]

and

\[
\int_S \chi x e^{-\nu x} \bar{f}(x) e^{U_{\delta}} \frac{a \delta^2 - |y_\xi(x)|^2}{(\delta^2 + |y_\xi(x)|^2)^2} dv_q = \frac{4\pi}{3\delta^2} (2a - 1) \bar{f}(\xi) + (a - 2) \frac{\pi}{3} \Delta_g \bar{f}(\xi) + O(\delta^\gamma)
\]

for \( a \in \mathbb{R} \).

We are now ready to establish the expansion of \( J_\lambda(U) \):

**Claim 4.2.** The following expansion does hold

\[
J_\lambda(U) = 2\pi k - \frac{\lambda |S|}{2} + 8\pi \lambda \psi_k(\xi, m) + O(\lambda^2 |\log \lambda|^2)
\]

in \( C(\Xi \times M) \) as \( \lambda \to 0^+ \).
Proof: First, let us consider the term
\[\int_S |\nabla V|^2 dv_g = \int_S V(-\Delta_g V) dv_g = \sum_{j,l=1}^k m_j m_l \int_S \chi_j e^{-\varphi_j} e^{U_j} PU_j dv_g\]
in view of \(\int_S V dv_g = 0\). Since by (1.5) and (2.1)
\[\int_S \chi_j e^{-\varphi_j} e^{U_j} G(x,\xi_l) dv_g = \int_S (-\Delta_g P U_j)G(x,\xi_l) dv_g = P U_j(\xi_l)\tag{4.2}\]
for all \(j,l = 1, \ldots, m\), by Lemmata 2.1, 4.1 and (4.2) we have that for \(l = j\)
\[\int_S \chi_j e^{-\varphi_j} e^{U_j} P U_j dv_g\]
\[= \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[\chi_j(U_j - \log(8\mu_j^2 \bar{\varepsilon}_j^2)) + 8\pi H(x,\xi_j) + O(\bar{\varepsilon}_j^2 |\log \varepsilon_j|)\right] dv_g\]
\[= \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[\chi_j \log \left(\frac{|y_j(x)|^4}{(\mu_j^2 \bar{\varepsilon}_j^2 + |y_j(x)|^2)^2}\right) + 8\pi G(x,\xi_j)\right] dv_g + O(\bar{\varepsilon}_j^2 |\log \varepsilon_j|)\]
\[= 8 \int_{B_{2r_0/\mu_j \varepsilon_j}(0)} \frac{\chi_j^2(\mu_j \varepsilon_j |y_j|)}{(1 + |y|^2)^2} \log \left(\frac{|y|^4}{(1 + |y|^2)^2}\right) dy + 8\pi P U_j(\xi_j) + O(\bar{\varepsilon}_j^2 |\log \varepsilon_j|)\]
\[= -16\pi - 32\pi \log \mu_j \varepsilon_j + 64\pi^2 H(\xi_j, \xi_j) + O(\bar{\varepsilon}_j^2 |\log \varepsilon_j|)\]
in view of
\[\int_{\mathbb{R}^2} \frac{dy}{(1 + |y|^2)^2} \log \left(\frac{|y|^4}{(1 + |y|^2)^2}\right) = 2\pi \int_0^\infty \frac{ds}{(1 + s)^2} \log \frac{s}{1 + s} = -2\pi \int_0^\infty \frac{ds}{(1 + s)^2} = -2\pi\]
by means of an integration by parts. Similarly, by Lemmata 2.1, 4.1 and (4.2) we have for \(l \neq j\)
\[\int_S \chi_j e^{-\varphi_j} e^{U_j} P U_j dv_g = \int_S \chi_j e^{-\varphi_j} e^{U_j} \left[8\pi G(x,\xi_l) + O(\bar{\varepsilon}_l^2 |\log \varepsilon_l|)\right] dv_g\]
\[= 64\pi^2 G(\xi_l, \xi_j) + O(\bar{\varepsilon}_l^2 |\log \varepsilon_l| + \bar{\varepsilon}_j^2 |\log \varepsilon_j|).\]
By using the definition of \(\mu_j\) and \(\varepsilon_j\) and summing up the two previous expansions, for the gradient term we get that
\[\frac{1}{2} \int_S |\nabla U|^2 dv_g = \frac{\lambda}{2} \int_S |\nabla V|^2 dv_g\]
\[= 2\pi - 8\pi \lambda (1 - \log 8) \sum_{j=1}^k m_j^2 + 8\pi \lambda \sum_{j=1}^k m_j^2 \log(2m_j^2) - 32\pi^2 \lambda \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j)\]
\[-32\pi^2 \lambda \sum_{i \neq j} m_j m_i G(\xi_i, \xi_j) + \lambda \sum_{j=1}^k O(\bar{\varepsilon}_j^2 |\log \varepsilon_j|)\].

Let us now expand the potential term in \(J(\mathcal{U})\). By Lemma 2.1 for any \(j = 1, \ldots, k\) we find that \(P U_j = O(|\log |\log \varepsilon_j||) = O(|\log \lambda|)\), in \(B_{r_0}(\xi_j) \setminus B_{\log \varepsilon_j}(\xi_j)\). Recall that \(P U_j = O(1)\) in \(S \setminus \bigcup_{j=1}^k B_{r_0}(\xi_j)\) for each \(j = 1, \ldots, k\). Hence, we have that \(V = O(|\log \lambda|)\), in \(S \setminus \bigcup_{j=1}^k B_{\log \varepsilon_j}(\xi_j)\) and also,
\[\int_S e^{\lambda V^2} dv_g = \sum_{j=1}^k \int_{B_{\log \varepsilon_j}(\xi_j)} e^{\lambda V^2} dv_g + |S| + O(\lambda |\log \lambda|^2)\]
Now, we write
\[ \int_{B_{|\log \varepsilon_j|^{-1}}(\xi_j)} e^{\lambda V^2} dv_g = \left[ \int_{A_{|\log \varepsilon_j|^{-1}, \delta \sqrt{\varepsilon_j}}(\xi_j)} + \int_{A_{\sqrt{\varepsilon_j}, \log \varepsilon_j}(\xi_j)} + \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} \right] e^{\lambda V^2} dv_g \]
In \( A_{|\log \varepsilon_j|^{-1}, \delta \sqrt{\varepsilon_j}}(\xi_j) \), we know that uniformly \( V(x) = -4m_j \log |y_{\xi_j}(x)| + O(1) \). Thus, we find that
\[ \int_{A_{|\log \varepsilon_j|^{-1}, \delta \sqrt{\varepsilon_j}}(\xi_j)} e^{\lambda V^2} dv_g = \int_{A_{|\log \varepsilon_j|^{-1}, \delta \sqrt{\varepsilon_j}}(\xi_j)} e^{\lambda m_j^2 \left[ 16 \log^2 |y_{\xi_j}(x)| + O(\log |y_{\xi_j}(x)|) \right]} dv_g \]
\[ = O \left( \int_{B_{|\log \varepsilon_j|^{-1}}(0)} e^{16 \lambda m_j^2 \log |y|} dy \right) = O(\lambda), \]
in view of \( e^\varepsilon_j = O(1), \lambda m_j^2 \log |y| = O(1) \) in the considered region and
\[ \int_{B_{|\log \varepsilon_j|^{-1}}(0) \setminus B_{\delta \sqrt{\varepsilon_j}}(0)} e^{16 \lambda m_j^2 \log |y|} dy = 2\pi \int_{\delta \sqrt{\varepsilon_j}}^{\log(\delta \log \varepsilon_j)} e^{16 \lambda m_j^2 \log^2 s} s \, ds \]
\[ = 2\pi \int_{\log(\delta \log \varepsilon_j)}^{\log(\delta \log \varepsilon_j)} e^{2t + 16 \lambda m_j^2 t^2} dt \quad (t = \log s) \]
\[ = O \left( \int_{\log(\delta \log \varepsilon_j)} e^t \, dt \right) = O(\lambda). \]
Now, we shall use that in \( B_{\delta \varepsilon_j}(\xi_j) \) it holds that \( e^{\lambda V^2} = 2m_j^2 e^{-\varepsilon_j} e^{w_j + \theta_j + \lambda m_j^2 (w_j + \theta_j)^2} \) and \( \lambda w_j = O(1) \) uniformly. Furthermore, we know that \( w_j(1 + \lambda m_j^2 w_j) \leq \frac{3}{4} w_j + \beta \), where \( \beta \) is a constant in \( A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j| \). Hence, we obtain that
\[ \int_{A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|} e^{\lambda V^2} = O \left( \int_{A_{\delta \sqrt{\varepsilon_j}, \delta \varepsilon_j}|\log \varepsilon_j|} e^{-\varepsilon_j} e^{\lambda m_j^2 w_j^2} \right) = O(\lambda). \]
Also, it follows that
\[ \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} e^{\lambda V^2} dv_g = 2m_j^2 \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} e^{-\varepsilon_j} e^{w_j + \theta_j + O(\lambda \log \lambda)} dv_g \]
\[ = 2m_j^2 \left[ 8\pi + O(\lambda \log \lambda^2) \right], \]
in view of \( w_j = O(\log \lambda) \) in \( B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j) \) and
\[ \int_{B_{\delta \varepsilon_j}|\log \varepsilon_j|(\xi_j)} \varepsilon_j^{-2} e^{w_j + \theta_j} = 8\pi + O \left( \lambda + \sum_{j=1}^{k} \varepsilon_j^2 |\log \varepsilon_j| \right). \]
Therefore, we conclude that
\[ \int_{S} e^{\lambda V^2} dv_g = 16\pi \sum_{j=1}^{k} m_j^2 + |S| + O(\lambda \log \lambda^2) \quad (4.3) \]
and consequently,
\[ J_\lambda(U) = 2\pi k + 8\pi \lambda (\log(16 - 1) - \sum_{j=1}^{k} m_j^2 + 8\pi \lambda \sum_{j=1}^{k} m_j^2 \log(m_j^2) - 32\pi^2 \lambda \sum_{j=1}^{k} m_j^2 H(\xi_j, \xi_j) \]
\[ - 32\pi^2 \lambda \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j) + \lambda \sum_{j=1}^{k} O(e_j^2 |\log \varepsilon_j|) - \frac{\lambda}{2} \left[ 16\pi \sum_{j=1}^{k} m_j^2 + |S| + O(\lambda \log \lambda^2) \right]. \]
Hence, we have that

\[ \text{Proof:} \]

This completes the proof.

In order to expand the derivatives \( \partial_{(\xi_q)} \mathcal{F}_\lambda \) and \( \partial_{m_l} \mathcal{F}_\lambda \), and get some estimates for \( \partial_{(\xi_q)} \phi \) and \( \partial_{m_l} \phi \), we have to expand \( \partial_{(\xi_q)} V \) and \( \partial_{m_l} V \) for \( q = 1, 2 \) and \( l = 1, \ldots, k \). Let us notice that from the definition of \( PU_j \) and similar ideas to obtain the expansion (2.6), we have that the derivatives \( \partial_{(\xi_q)} PU_j \), for \( q = 1, 2 \) and \( \partial_{m_l} PU_j \) for \( j, l = 1, \ldots, k \) expand as follows

\[
\partial_{(\xi_q)} PU_j(x) = \delta_j \partial_{(\xi_q)} x_j [U_j - \log(8\mu_j^2 \varepsilon_j^2)] + x_j \partial_{(\xi_q)} [U_j - \log(8\mu_j^2 \varepsilon_j^2)] + \delta_j 8\pi \partial_{(\xi_q)} H(x, \varepsilon_j) + O(\varepsilon_j^2 |\log \varepsilon_j|) \tag{4.4}
\]

and

\[
\partial_{m_l} PU_j(x) = \chi_j \partial_{m_l} [U_j - \log(8\mu_j^2 \varepsilon_j^2)] + O(\varepsilon_j^2 |\log \varepsilon_j|), \tag{4.5}
\]

uniformly in \( S \). Let us stress that \( \mu_j = \mu_j(\xi, m) \) and \( \varepsilon_j = \varepsilon_j(m_j) \). Furthermore, from the definition of \( \varepsilon_l \) we get

\[
\partial_{m_l} \varepsilon_l = \frac{\varepsilon_l}{4} \left( \frac{1}{\lambda m_l^2} + \frac{4}{m_l^4} \right) - \frac{\varepsilon_l}{m_l} [-2 \log \varepsilon_l + \log(2m_l^2) + 1]. \tag{4.6}
\]

Hence, we have that uniformly in \( S \)

\[
\partial_{(\xi_q)} V(x) = -2 m_l \partial_{(\xi_q)} \chi_l \log(\mu_l^2 \varepsilon_l^2 + |y_l(x)|^2) - m_l \chi_l \frac{4 \mu_l \partial_{(\xi_q)} \mu_l \varepsilon_l^2 + 2 \partial_{(\xi_q)} (|y_l(x)|^2)}{\mu_l^2 \varepsilon_l^2 + |y_l(x)|^2} + 8 \pi m_l \partial_{(\xi_q)} H(x, \varepsilon_l) - \sum_{j \neq l} m_j \chi_j \frac{4 \mu_j \partial_{(\xi_q)} \mu_j \varepsilon_j^2}{\mu_j^2 \varepsilon_j^2 + |y_j(x)|^2} + \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|) \tag{4.7}
\]

and

\[
\partial_{m_l} V(x) = -2 \chi_l \log(\mu_l^2 \varepsilon_l^2 + |y_l(x)|^2) + 8 \pi H(x, \varepsilon_l) - m_l \chi_l \frac{4 \mu_l \partial_{m_l} \mu_l \varepsilon_l^2 + 4 \mu_l^2 \varepsilon_l \partial_{m_l} \varepsilon_l}{\mu_l^2 \varepsilon_l^2 + |y_l(x)|^2} - \sum_{j \neq l} m_j \chi_j \frac{4 \mu_j \partial_{m_l} \mu_j \varepsilon_j^2}{\mu_j^2 \varepsilon_j^2 + |y_j(x)|^2} + \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|). \tag{4.8}
\]

Claim 4.3. The following expansion does hold

\[
\partial_{m_l} [J_{\lambda}(U)] = 8 \pi \lambda \partial_{m_l} \psi_k(\xi, m) + O(\lambda^2 |\log \lambda|^2) \tag{4.9}
\]

in \( C(\Xi \times M) \) as \( \lambda \to 0^+ \).

**Proof:** Notice that

\[
\partial_{m_l} [J_{\lambda}(U)] = \lambda \int_S \left[ \Delta_g V + \lambda V e^{\lambda V} \right] \partial_{m_l} V \, dv_g.
\]

Hence, we have that

\[
\int_S \Delta_g V \, dv_g = - \sum_{i=1}^k m_i \int_S \chi_i e^{-\varphi_i} e^{U_i} \partial_{m_l} V
\]

in view of \( \int_S \partial_{m_l} V = 0 \). For \( i \neq l \), by using Lemma 4.1 and (4.8), we find that

\[
\int_S \chi_i e^{-\varphi_i} e^{U_i} \partial_{m_l} V \, dv_g = \int_S \chi_i e^{-\varphi_i} e^{U_i} \left[ 8 \pi G(x, \xi_l) - m_i \chi_i \frac{4 \mu_l \partial_{m_l} \mu_l \varepsilon_l^2}{\mu_l^2 \varepsilon_l^2 + |y_l(x)|^2} + \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|) \right] = 64 \pi^2 G(\xi_l, \xi_l) - 16 \pi m_i \frac{\partial_{m_l} \mu_l}{\mu_l} + \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|^2) = \sum_{j=1}^k O(\varepsilon_j^2 |\log \varepsilon_j|^2),
\]
in view of $\chi_j \equiv 0$ in $B_{2r_0}(\xi_i)$ with $j \neq i$ and from (2.7)

$$2\frac{\partial m_0\mu_j}{\mu_j} = 8\pi m_j^{-1}G(\xi_i, \xi_j) \quad \text{with } j \neq l.$$  

(4.10)

Also, using again (4.8), we find that

$$\int_S \chi_i e^{-\omega_i} e^{U_i} \partial_{m_1} V d\nu_g = \int_S \chi_i e^{-\omega_i} e^{U_i} \left[ -2\chi_i \log(\mu_i^2 \varepsilon_i^2 + |y_{\varepsilon_i}(x)|^2) + 8\pi H(x, \xi_i) - m_i \chi_i \frac{2\partial m_i(\mu_i^2 \varepsilon_i^2)}{\mu_i^2 \varepsilon_i^2 + |y_{\varepsilon_i}(x)|^2} + \sum_{j=1}^k O\left(\varepsilon_j^2 \log \varepsilon_j|\varepsilon_j|^2\right) \right]$$

$$= -16\pi - 32\pi \log(\mu_i \varepsilon_i) + 64\pi^2 H(\xi_i, \xi_i) - 8\pi m_i \frac{\partial_m(\mu_i^2 \varepsilon_i^2)}{\mu_i^2 \varepsilon_i^2}$$

$$+ \sum_{j=1}^k O\left(\varepsilon_j^2 \log \varepsilon_j|\varepsilon_j|^2\right).$$

From the definition (2.7)-(2.8), we get that

$$2\frac{\partial m_0\mu_i}{\mu_i} = -\frac{4}{m_i} \frac{8\pi}{m_i^2} \sum_{i \neq l} m_i G(\xi_i, \xi_l) \quad \text{and} \quad -\frac{4}{\varepsilon_i} \partial_{m_i} \varepsilon_i = -\frac{1}{\lambda m_i^3} - \frac{4}{m_j}$$  

(4.11)

so that, we conclude that

$$\int_S \Delta_g V \partial_{m_1} V d\nu_g = -16\pi \log 2m_i - 16\pi m_i \log(2m_i^2) + 64\pi^2 m_i H(\xi_i, \xi_i)$$

$$+ 64\pi^2 \sum_{j=1, j \neq l}^k m_j G(\xi_j, \xi_l) + O(\lambda|\log \lambda|).$$

On the other hand, combining arguments to deduce (2.22) and (4.3), using Lemma 4.1 and the derivatives (4.8) and (4.10)-(4.11), we find that

$$\int_S \lambda^2 e^{\lambda V^2} \partial_{m_1} V d\nu_g = 16\pi m_i + O(\lambda|\log \lambda|).$$

Thus, we conclude (4.9).  

\[\square\]

**Claim 4.4.** The following expansion does hold

$$F_\lambda(\xi, m) = J_\lambda(U) + \tilde{\theta}_\lambda(\xi, m)$$

(4.12)

in $C(\Xi \times \mathcal{M})$ and $C^1(\mathcal{M})$ as $\lambda \to 0^+$, where the term $\tilde{\theta}_\lambda(\xi, m)$ satisfies

$$|\tilde{\theta}_\lambda(\xi, m)| + \sum_{l=1}^k \lambda \left|\partial_{m_l} \tilde{\theta}_\lambda(\xi, m)\right| = O(\lambda^3)$$

(4.13)

as $\lambda \to 0^+$ uniformly for points $(\xi, m) \in \Xi \times \mathcal{M}$.

**Proof:** Taking into account that $DJ_\lambda(U + \tilde{\phi})(\tilde{\phi}) = 0$, a Taylor expansion, $\sqrt{\lambda} V = U$ and

$$\sqrt{\lambda} \phi(\xi, m) = \phi(\xi, m),$$

the definition of $K$ and (3.6) gives us

$$J_\lambda(U + \tilde{\phi}) - J_\lambda(U) = -\int_0^1 D^2 J_\lambda(U + t\tilde{\phi})(\tilde{\phi}, \tilde{\phi}) \, dt,$$

$$= -\lambda \int_0^1 \left( \int_S [R + N(\phi)] \phi + \int_S [K - f^j(V + t\phi)] \phi^2 \right) \, dt.$$  

(4.14)
Therefore, we get
\[ J_\lambda(U + \tilde{\phi}) - J_\lambda(U) = O(\lambda^3), \]
since \( \|R\| \leq C \lambda, \|N(\phi)\| \leq C\|\lambda\|\phi\|_\infty + \|\phi\|_\infty^2 \), for some \( s \in (0, t) \)
\[ \|K - f'(V + t\phi)\| \leq \|f''(V + s\phi)\| \phi\|_\infty + \|K - f'(V)\| \leq C(\|\phi\|_\infty + \lambda) \]
and \( \|\phi\|_\infty \leq C\lambda \).

Let us differentiate with respect to \( \beta = m_l \). We use representation \((1.14)\) and differentiate directly under the integral sign, thus obtaining, for each \( l = 1, \ldots, k \)
\[ \partial_\beta [J_\lambda(U + \tilde{\phi}) - J_\lambda(U)] = -\lambda \int_0^1 \left( \int_S \{[\partial_\beta R + \partial_\beta N(\phi)] \phi + [R + N(\phi)] \partial_\beta \phi\} \right) t \, dt \]
\[ - \lambda \int_0^1 \left( \int_S \{[\partial_\beta K - \partial_\beta [f'(V + t\phi)]] \phi^2 + [K - f'(V + t\phi)] 2\phi \partial_\beta \phi\} \right) t \, dt. \]

Using Proposition \(3.2\) and the computations in the Appendix B, we get that
\[ |\partial_m [J_\lambda(U + \tilde{\phi}) - J_\lambda(U)]| \leq C\lambda \left( [\lambda \log \varepsilon_l + \lambda^2 \log \varepsilon_l + \lambda\|\partial_m \phi\|_\infty] \|\phi\|_\infty \right. \]
\[ + \left[\lambda + \lambda\|\phi\|_\infty + \|\phi\|_\infty^2 \| \partial_m \phi\|_\infty + \lambda \log \varepsilon_l ||\phi||^2_\infty + [\|\phi\|_\infty + \lambda ] ||\phi||_\infty \| \partial_m \phi\|_\infty \right). \]

Thus, we conclude
\[ \partial_m [J_\lambda(U + \tilde{\phi}) - J_\lambda(U)] = O(\lambda^3 \log \varepsilon_l) = O(\lambda^2), \quad l = 1, \ldots, k \]
Now, taking \( \partial_\lambda (\xi, m) = F_\lambda(\xi, m) - J_\lambda(U) \), we have shown \(4.13\) as \( \lambda \to 0^+ \). The continuity in \((\xi, m)\) of all these expressions is inherited from that of \( \phi \) and its derivatives in \( \xi \) and \( m \) in the \( L^\infty \) norm.

Now, we are going to study the derivatives of \( F_\lambda \) with respect to \( \beta = (\xi)_q \) with \( q = 1, 2 \) and \( l = 1, \ldots, k \). Due to the estimates \(3.3\) given in Proposition \(3.2\), we have to address this expansion in a different way. We shall use similar ideas first presented in \(20\) and also used in \(9\).

**Claim 4.5.** The following expansion does hold
\[ \partial_{(\xi)_q} F_\lambda(\xi, m) = 8\pi \lambda \partial_{(\xi)_q} \psi_k(\xi, m) + O(\lambda^2 |\log \lambda|^2) \quad (4.15) \]
in \( C(\Xi \times M) \) as \( \lambda \to 0^+ \).

**Proof:** Let us differentiate the function \( F_\lambda(\xi, m) \) with respect to \((\xi)_q \) with \( q = 1, 2 \) and \( l = 1, \ldots, k \). Since \( \sqrt{\lambda} V(\xi, m) = U(\xi, m) \) and \( \sqrt{\lambda} \phi(\xi, m) = \phi(\xi, m) \), we can differentiate directly \( J_\lambda(\sqrt{\lambda} V + \phi) \) (under the integral sign), so that integrating by parts we get
\[ \partial_{(\xi)_q} F_\lambda(\xi, m) = -\lambda \int_S \left[ \Delta_g(V + \phi) + \lambda(V + \phi)e^{\lambda(V + \phi)^2} \right] \partial_{(\xi)_q} V + \partial_{(\xi)_q} \phi \right] dv_g 
\[ = -\lambda \sum_{i=0}^k \sum_{j=1}^k c_{ij} \int_S \Delta_g P Z_{ij} \partial_{(\xi)_q} V + \partial_{(\xi)_q} \phi \right] dv_g, \]

since \( \int_S (\partial_{(\xi)_q} V + \partial_{(\xi)_q} \phi) dv_g = 0 \). From the orthogonal conditions we find that for \( j \neq l \)
\[ c_{ij} \int_S \Delta_g P Z_{ij} \partial_{(\xi)_q} \phi dv_g = -c_{ij} \int_S \partial_{(\xi)_q} [\Delta_g P Z_{ij}] \phi dv_g = O \left( \max_{ij} |c_{ij}| \|\phi\|_\infty \right) = O(\lambda^2) \]
in view of \( \int \rho(x) \, dv_g \leq C, \| \partial_{(\xi)_q} \Delta_g PZ_{ij} \|_* \leq C \) for \( j \neq l \) and \( 3.7 \). For \( j = l \), we compare \( \partial_{(\xi)_q} \Delta_g PZ_{il} \) with derivatives \( \partial_{x_q} \Delta_g PZ_{il} \) to get

\[
\int_S \Delta_g PZ_{il} \partial_{(\xi)_q} \phi \, dv_g = - \int_S \partial_{(\xi)_q} [\Delta_g PZ_{il}] \phi \, dv_g = \int_S \partial_{x_q} [\Delta_g PZ_{il}] \phi \, dv_g + O(\lambda)
\]

Thus, integrating by parts we deduce that

\[
\sum_{i=0}^2 \sum_{j=1}^k c_{ij} \int_S \Delta_g PZ_{ij} \partial_{(\xi)_q} \phi \, dv_g = - \sum_{i=0}^2 \int_S \Delta_g PZ_{il} \partial_{x_q} \phi \, dv_g + O(\lambda^2)
\]

\[
= - \int_{B_{2r_0}(\xi)} \left[ \Delta_g (V + \phi) + \lambda (V + \phi)e^{\lambda(V + \phi)} \right] \partial_{x_q} \phi \, dv_g + O(\lambda^2).
\]

On the other hand, from \( 4.4 \), we obtain that for \( j \neq l \) in \( B_{2r_0}(\xi) \)

\[
\partial_{(\xi)_q} V(x) = 8\pi m_l \partial_{(\xi)_q} G(x, \xi_l) - m_j \chi \frac{4\mu_j \partial_{(\xi)_q} \mu_j \varepsilon_j^2}{\mu_j^2 \varepsilon_j^2 + |y_{\xi_j}(x)|^2} + \sum_{i=1}^k O \left( \varepsilon_j^2 |\log \varepsilon_i| \right)
\]

and consequently

\[
\int_S \Delta_g PZ_{ij} \partial_{(\xi)_q} V \, dv_g = - \int_{B_{2r_0}(\xi)} \chi_j e^{-\varepsilon_j^2} \varepsilon_j^{-2} e^{w_j} Z_{ij} \left[ 8\pi m_l \partial_{(\xi)_q} G(x, \xi_l) - m_j \chi \frac{4\mu_j \partial_{(\xi)_q} \mu_j \varepsilon_j^2}{\mu_j^2 \varepsilon_j^2 + |y_{\xi_j}(x)|^2}
\]

\[
+ \sum_{i=1}^k O \left( \varepsilon_j^2 |\log \varepsilon_i| \right) \right) \, dv_g
\]

From the definition of \( Z_{ij} \) it follows that for \( j \neq l \)

\[
\int_S \Delta_g PZ_{ij} \partial_{(\xi)_q} V \, dv_g = \begin{cases} \mathcal{O}(1) & \text{if } i = 0, \\ \mathcal{O}(\varepsilon_j) & \text{if } i = 1, 2. \end{cases}
\]

For \( j = l \), we compare \( \partial_{(\xi)_q} V \) with derivatives \( \partial_{x_q} V \) to get

\[
\int_S \Delta_g PZ_{il} \partial_{(\xi)_q} V \, dv_g = - \int_S \Delta_g PZ_{il} \partial_{x_q} V \, dv_g + O(\lambda).
\]

Hence, taking into account that \( |c_{ij}| \leq C \lambda \) for all \( i = 0, 1, 2 \) and \( j = 1, \ldots, k \), we obtain that

\[
\sum_{i=0}^2 \sum_{j=1}^k c_{ij} \int_S \Delta_g PZ_{ij} \partial_{(\xi)_q} V \, dv_g = \sum_{i=0}^2 \int_S \Delta_g PZ_{il} \partial_{(\xi)_q} V \, dv_g + O(\lambda)
\]

\[
= - \sum_{i=0}^2 \int_S \Delta_g PZ_{il} \partial_{x_q} V \, dv_g + O(\lambda)
\]

\[
= - \int_{B_{2r_0}(\xi)} \left[ \Delta_g (V + \phi) + \lambda (V + \phi)e^{\lambda(V + \phi)} \right] \partial_{x_q} V + O(\lambda).
\]

Therefore, denoting \( v_\xi = V + \phi \) we get that

\[
\partial_{(\xi)_q} \mathcal{F}_\lambda(\xi, m) = \lambda \int_{B_{2r_0}(\xi)} \left[ \Delta_g v_\xi + \lambda v_\xi e^{\lambda v_\xi} \right] \partial_{x_q} v_\xi \, dv_g + O(\lambda)
\]
To be more precise, we will need to take \( \delta r \) small.

Proof: Direct computations shows that \((\text{Lemma A.1.})\)

Therefore, taking into account the expansions \((4.1), (4.9), (4.12)\) and \((4.15)\) we conclude the proof of Proposition \((3.4)\).

Appendix A: The linear theory

In this section, we will study the linearized operator under suitable orthogonality conditions. Throughout the main part of this section we only assume that the numbers \(\mu_j, j = 1, \ldots, k\) satisfy \(C_0^{-1} \leq \mu_j \leq C_0\) for all \(j = 1, \ldots, k\) independently of \(\lambda\) and that the points \(\xi_j \in \Sigma, j = 1, \ldots, k\) are uniformly separated from each other, namely, \(\xi = (\xi_1, \ldots, \xi_k) \in \Xi\).

For \(\mu, \varepsilon, m > 0\) define the function

\[
\rho_{\mu, \varepsilon, m}(y) = \lambda \left( \frac{|r_j|}{|\log r_j|} \right) \left( 1 + |w_{\mu}(|\frac{y}{\varepsilon}|)| + |w_{\mu}(|\frac{y}{\varepsilon}|)^2 \right) e^{-2\frac{w_{\mu}(\frac{r_j}{\varepsilon})}{\varepsilon}},
\]

with \(y \in \mathbb{R}^2\), so that, \(\rho_j(x) = \rho_{\mu_j, \varepsilon, m_j}(y_{\xi_j}(x)) \) for \(x \in B_{2r_0}(\xi_j)\), with \(\rho_j\) defined in \((2.12)\). First, we will prove the following result.

Lemma A.1. There exist positive constants \(C, \varepsilon_0 > 0\) with \(C = C(\delta, r)\) such that for all \(0 < \varepsilon < \varepsilon_0\) the solution \(\psi\) to problem

\[
-\Delta \psi = \rho_{\mu, \varepsilon, m}, \quad \delta \varepsilon < |y| < r,
\]

satisfies the estimate \(\|\psi\|_\infty \leq C\). Furthermore, \(C(\delta, r)\) could be smaller if we choose \(\delta\) large and \(r\) small.

To be more precise, we will need to take \(\delta\) large and \(r\) small enough so that \(2C(\delta, r) + r^2 < 1\).

Proof: Since \(\rho_{\mu, \varepsilon, m}\) is radial, \(\psi(y) = \psi(|y|)\). If \(\varphi(t) = \psi(|y|)\) then for \(t \in [\log(\delta \varepsilon), \log r]\) we study

\[
-\varphi''(t) = e^{2t} \rho_{\mu, \varepsilon, m}(e^t) := g_0(t), \quad \varphi(\log(\delta \varepsilon)) = 0, \quad \varphi(\log r) = 0.
\]

Direct computations shows that

\[
\varphi(t) = \frac{t - \log(\delta \varepsilon)}{\log \frac{r}{\delta \varepsilon}} \int_{\log(\delta \varepsilon)}^{\log r} \int_{\log(\delta \varepsilon)}^{s} g_0(\tau) d\tau ds - \int_{\log(\delta \varepsilon)}^{t} \int_{\log(\delta \varepsilon)}^{s} g(\tau) d\tau ds.
\]

Notice that \(\varphi\) is concave and its maximum \(t_0\) satisfies

\[
\varphi'(t_0) = \frac{1}{\log \frac{r}{\delta \varepsilon}} \int_{\log(\delta \varepsilon)}^{\log r} \int_{\log(\delta \varepsilon)}^{s} g(\tau) d\tau ds - \int_{\log(\delta \varepsilon)}^{t_0} \int_{\log(\delta \varepsilon)}^{s} g(\tau) d\tau ds = 0
\]
and hence,
\[ \varphi(t_0) = \int_{\log(\delta \lambda)}^{t_0} g_0(s)(s - \log(\delta \varepsilon)) \, ds = \int_{\log(\lambda)}^{\log r} g(s)(\log r - s) \, ds. \]

Thus, we deduce that for all \( t \in [\log(\delta \lambda), \log r] \)
\[ 0 \leq \varphi(t) \leq \varphi(t_0) \leq \max \left\{ \int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} g_0(s)(s - \log(\delta \varepsilon)) \, ds, \int_{\log(\delta \lambda)}^{\log r} g(s)(\log r - s) \, ds \right\}. \]

We estimate every integral in the following way
\[
\int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} g_0(s)(s - \log(\delta \varepsilon)) \, ds \leq C \left[ \int_{\log(\delta \lambda)}^{\log(2\delta \varepsilon)} e^{2\varepsilon e^{-2s}(1 + |\log \varepsilon - s| + |\log \varepsilon - s|^2)}(s - \log(\delta \varepsilon)) \, ds \\
+ \int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} \varepsilon \{1 + s\} e^{-s} + \lambda^{-1} \varepsilon^{-2s} \} (s - \log(\delta \varepsilon)) \, ds \right] \\
\leq C \int_{\log(\delta \lambda)}^{\log(2\delta \varepsilon)} e^{-2(1 + |t| + |t|^2)(t - \log \delta) \, dt} \\
+ \int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} \{(1 + t + \log \varepsilon) e^{-t} + \lambda^{-1} e^{-2t} \} (t - \log \delta) \, dt \\
\leq C_1(\delta)[C_2(\varepsilon) + 1] \quad \text{(with } t = s - \log \varepsilon \text{)}
\]

with \( C_1(\delta), C_2(\varepsilon) \to 0 \) as \( \delta \to +\infty \) and \( \varepsilon \to 0 \), in view of (A.1),
\[
\left| w_\mu \left( \frac{e^{\varepsilon \lambda}}{e} \right) \right| = |\log(8\mu^2) - 2\log(1 + \mu^2 e^{-2s} + 4(\log \varepsilon - s))| = O(1 + |\log \varepsilon - s|),
\]
for \( s \in [\log(\delta \lambda), \log(2\delta \varepsilon)| \log \varepsilon|^2] \), \( e^{-2e^{\mu} \varepsilon}(\frac{e^{\varepsilon \lambda}}{e}) = O(e^{2e^{-4s}}) \), for \( s \in [\log(\delta \lambda), \log(\delta \sqrt{\lambda})] \) and \( e^{\mu} \varepsilon}(\frac{e^{\varepsilon \lambda}}{e}) = O(e^{4e^{-s}}(1 + |s|)), \) for \( s \in [\log(\delta \lambda)| \log \varepsilon|^2], \log(2\delta \sqrt{\varepsilon}) \). And for the second integral
\[
\int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} g_0(s)(\log r - s) \, ds \leq C \int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} \left[ (1 + |s|) e^{2s + 16\lambda m^2 s^2 + \lambda^{-1} e^{-2s}} \right] (\log r - s) \, ds \\
\leq C \int_{\log(\delta \lambda)}^{\log(\sqrt{\lambda})} \left[ e^{s}(1 + |s| + |s|^2) + \lambda^{-1} e^{2s} (\log r - s) \right] \, ds \\
\leq C_3(r)[1 \, + \, C_2(\varepsilon)] \, + \, C_1(\delta)C_2(\varepsilon)[1 \, + \, \log r] 
\]
with \( C_3(r) \to 0 \) as \( r \to 0 \), in view of (A.1), \( e^{-2e^{\mu} \varepsilon}(\frac{e^{\varepsilon \lambda}}{e}) = O(e^{2e^{-4s}}) \), for \( s \in [\log(\delta \sqrt{\varepsilon}), \log r] \) and \( e^{-2e^{\mu} \varepsilon}(\frac{e^{\varepsilon \lambda}}{e}) + \lambda m^2 e^{\mu} \varepsilon(\frac{e^{\varepsilon \lambda}}{e}) = O(e^{16\lambda m^2 s^2}) \), and \( e^{2s + 16\lambda^2 m^2 s^2} = O(e^s) \), for \( s \in [\log(\delta \sqrt{\varepsilon}), \log r] \). 

We are now ready for

**Proof (of Proposition 3.1):** The proof of estimate (3.4) consists of several steps. Let assume the opposite, namely, the existence of sequences \( \lambda_n \to 0 \), points \( \xi^0 = (\xi^0_1, \ldots, \xi^0_k) \in \Xi \), numbers \( m^n \) with \( m^n = (m^n_1, \ldots, m^n_k) \in M \), \( \mu^n_j, \varepsilon^n_j \) and \( e^n_i \), functions \( h_n \) with \( ||h_n||_s \to 0 \) as \( n \to +\infty \), \( \phi_n \) with \( ||\phi_n||_\infty = 1 \), and
\[
\begin{align*}
L(\phi_n) &= h_n + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} c^n_i \Delta y P \Sigma^n_{ij}, \quad \text{in } S, \\
\int_S \phi_n \Delta y P \Sigma^n_{ij} dv_y &= 0, \quad \text{for all } i = 0, 1, 2, j = 1, \ldots, m, \quad \int_S \phi_n dv_y = 0.
\end{align*}
\]
Without loss of generality, we assume that \( \xi^0_j \to \xi^*_j \), \( m^n_j \to m^*_j \), \( \mu^n_j \to \mu^*_j \) as \( n \to +\infty \) and \( \xi^* = (\xi^*_1, \ldots, \xi^*_k) \in \Xi m^* = (m^*_1, \ldots, m^*_k) \in M \). First, we have the following fact.
Claim A.2. There exists a constant $C > 0$ independent of $n$ such that for all $i = 0, 1, 2$, $j = 1, \ldots, k$ it holds $|c_{ij}| \leq C \left( \|h_n\|_* + \|\phi_n\|_\infty \right)$ as $n \to +\infty$.

Proof: For notational purpose we omit the index $n$. To estimate the values of the $c_{ij}$’s, test equation (A.2) against $PZ_{ij}$, $i = 0, 1, 2$ and $j = 1, \ldots, m$:

$$\int_S \phi(\Delta_g PZ_{ij} + KPZ_{ij}) \, dv_g = \int_S hPZ_{ij} \, dv_g + \sum_{p=0}^2 \sum_{q=1}^k c_{pq} \int_S \Delta_g PZ_{pq} PZ_{ij} \, dv_g.$$ 

Since for $j = 1, \ldots, k$ we have the following estimates in $C(S)$

$$PZ_{ij} = \chi_j Z_{ij} + O(\varepsilon_j), \quad i = 1, 2, \quad PZ_{0j} = \chi_j (Z_{0j} + 2) + O(\varepsilon_j^2 |\log \varepsilon_j|), \quad (A.3)$$

it readily follows that

$$\int_S \Delta_g PZ_{pq} PZ_{ij} \, dv_g = -\frac{32\pi}{3} \delta_{pq} \delta_{ij} + O(\varepsilon_j), \quad (A.4)$$

where the $\delta_{ij}$’s are the Kronecker’s symbols. Furthermore, we find that

$$\int_S \phi \hat{L}(PZ_{0j}) \, dv_g = \int_S \phi \left( \chi_j \Delta_g Z_{0j} + \sum_{l=1}^k \chi_l e^{-\varphi_l} \varepsilon_l^{-2} e^{w_l} \left[ \chi_j (Z_{0j} + 2) + O(\varepsilon_j^2 |\log \varepsilon_j|) \right] \right) \, dv_g$$

$$= \int_S \phi \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \left( -Z_{0j} + \chi_j Z_{0j} + 2\chi_j \right) \, dv_g + O(\varepsilon_j^2 |\log \varepsilon_j| \|\phi\|_\infty)$$

$$= 2 \int_S \phi \chi_j^2 e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \, dv_g + O(\varepsilon_j^2 |\log \varepsilon_j| \|\phi\|_\infty)$$

and similarly for $i = 1, 2$

$$\int_S \phi \hat{L}(PZ_{ij}) \, dv_g = \int_S \phi \left( \chi_j \Delta_g Z_{ij} + \sum_{l=1}^k \chi_l e^{-\varphi_l} \varepsilon_l^{-2} e^{w_l} \left[ \chi_j Z_{ij} + O(\varepsilon_j) \right] \right) \, dv_g$$

$$= \int_S \phi \chi_j e^{-\varphi_j} \varepsilon_j^{-2} e^{w_j} \left( -Z_{ij} + \chi_j Z_{ij} \right) \, dv_g + O(\varepsilon_j \|\phi\|_\infty)$$

$$= O(\varepsilon_j \|\phi\|_\infty)$$

Hence, we get the estimates $|c_{0j}| \leq C \left( \|h\|_* + \|\phi\|_\infty + \varepsilon_j \sum_{p=0}^2 \sum_{q=1}^k |c_{pq}| \right)$ and

$$|c_{ij}| \leq C \left[ \|h\|_* + \varepsilon_j \|\phi\|_\infty + \varepsilon_j \sum_{p=0}^2 \sum_{q=1}^k |c_{pq}| \right], \quad \text{for } i = 1, 2. \quad (A.5)$$

Thus, the claim follows. \qed

Now, we prove the asymptotic behavior of $\phi_n$ on compact subsets of $S \setminus \{\xi_1, \ldots, \xi_m\}$.

Claim A.3. There holds $\phi_n \to 0$ as $n \to +\infty$ in $C^1$ uniformly over compact subsets of $S \setminus \{\xi_1, \ldots, \xi_m\}$. In particular, given any $2r_0 > r > 0$ we have

$$\|\phi_n\|_{L^\infty(S \setminus \cup_{j=1}^m B_r(\xi_j))} \to 0 \quad \text{as } n \to +\infty. \quad (A.6)$$
Proof: Note that for any $0 < r < r_0$ it holds that up to a subsequence as $n \to +\infty$
\[ c(\phi_n) = -\frac{1}{|S|} \int_S K(x)\phi_n(x) \, dv_g = -\frac{1}{|S|} \sum_{j=1}^k \int_{B_r(\xi_j)} e^{-\varphi_j}(\varepsilon_j^n)^{-2} e^{\mu_j} \phi_n \, dv_g + O(\varepsilon_j^n) = c_0 + o(1). \]

From Claim A.2 it readily follows that $\sum_{i=0}^2 \sum_{j=1}^k c_i^n \Delta_g PZ_{ij}^n = o(1)$ as $n \to +\infty$ in $S \setminus \cup_{j=1}^m B_r(\xi_j^n)$ for a given $r > 0$, in view of $\Delta_g PZ_{ij}^n = O(\varepsilon_j^n)$ in $S \setminus \cup_{j=1}^m B_r(\xi_j^n)$. Thus, we get
\[ \Delta_g \phi_n(x) = \sum_{j=1}^k O((\varepsilon_j^n)^2) + c_0 + o(1) \quad \text{uniformly for } x \in S \setminus \cup_{j=1}^m B_r(\xi_j^n). \]

Therefore, passing to a subsequence $\phi_n \to \phi^*$ as $n \to +\infty$ in $C^1$ sense over compact subsets of $S \setminus \{\xi_1^n, \ldots, \xi_m^n\}$. Since $|\phi^*(x)| \leq 1$ for all $x \in S \setminus \{\xi_1^n, \ldots, \xi_m^n\}$, it follows that $\phi^*$ can be extended continuously to $S$ so that $\phi^*$ satisfies $\Delta_g \phi^* = c_0$, in $S$ and $\int_S \phi^* \, dv_g = 0$ using dominated convergence. By, $\int_S \Delta_g \phi^* = 0$ we get that $c_0 = 0$. Therefore, $\phi^* \equiv 0$, and the claim follows. \hfill \Box

Claim A.4. For all $i = 0, 1, 2, j = 1, \ldots, k$ it holds that $c_{ij}^n \to 0$ as $n \to +\infty$.

Proof: The claim readily follows for $i = 1, 2$ and $j = 1, \ldots, k$ from Claim A.2 and the estimate in A.3 in view of $\|h\|_\ast = o(1)$. Let us refine the estimate for $c_{ij}, j = 1, \ldots, k$. It is clear that in $B_{r_0}(\xi_j)$
\[ \Delta_g \phi + e^{-\varphi_j} \varepsilon_j^{-2} e^{\mu_j} \phi + c(\phi) = h + \sum_{p=0}^{2} c_{pj} \Delta_g Z_{pj} - \frac{2}{|S|} \sum_{p=0}^{k} \sum_{q=1}^{k} c_{pq} \int_S \chi_g \Delta_g Z_{pq} \, dv_g. \]

Hence, integrating on $B_{r}(\xi_j)$ with $0 < r < r_0$ we find that as $n \to +\infty$
\[ \int_{B_r(\xi_j)} e^{-\varphi_j} \varepsilon_j^{-2} e^{\mu_j} \phi \, dv_g = -\int_{B_{r}(\xi_j)} \Delta_g \phi \, dv_g - c(\phi) |B_r(\xi_j)| + \int_{B_{r}(\xi_j)} h \, dv_g + \int_{B_{r}(\xi_j)} \Delta_g Z_{pj} \, dv_g - \frac{2}{|S|} \sum_{p=0}^{k} \sum_{q=1}^{k} \frac{c_{pq}}{\|h\|_\ast} |B_r(\xi_j)| \int_S \chi_g \Delta_g Z_{pq} \, dv_g \]
\[ = -\int_{\partial B_r(0)} \partial_r (\phi \circ y_j^{-1}) \, d\sigma + O \left( \|c(\phi)\| + \|h\|_\ast \varepsilon_j + \varepsilon_j \sum_{p=0}^{k} \sum_{q=1}^{k} \|c_{pq}\| \right) = o(1), \]

in view of Claims A.2 and A.3 (the convergence $\phi \to 0$ in $C^1$ sense). Thus, we obtain that
\[ c_{ij} = O \left( \int_{B_{r}(\xi_j)} e^{-\varphi_j} \varepsilon_j^{-2} e^{\mu_j} \phi \, dv_g \right) + \varepsilon_j^2 \|\phi\|_\infty + \|h\|_\ast + \varepsilon_j \sum_{p=0}^{k} \sum_{q=1}^{k} \|c_{pq}\| = o(1) \]
and the claim follows. \hfill \Box

We follow ideas shows in [14] to prove an estimate for $\phi_n$. For $y \in B_{r_0}(0)$, define $\hat{\phi}_{n,j}(y) = \phi_n(y_{\xi_j^{-1}}(y))$ and
\[ \hat{h}_n(y) = e^{\hat{\varphi}_j(y)} \left[ -c(\hat{\phi}_n) + h_n(y_{\xi_j}^{-1}(y)) + \sum_{p=0}^{k} \sum_{q=1}^{k} c_{pq} \Delta_g PZ_{pq}^n(y_{\xi_j}^{-1}(y)) \right], \]
so that
\[ \hat{L}_j(\hat{\phi}_{n,j}) := \Delta \hat{\phi}_{n,j} + \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 \varepsilon_j^2 + |y|^2)^2} \hat{\phi}_{n,j} = \hat{h}_n. \]
Let us fix such a number $\delta > 0$ which we may take larger whenever it is needed and a small $0 < r < r_0$. Now, let us consider the “annulus norm” and “boundary annulus norm”

$$
\|\phi\|_a = \|\phi\|_{L^\infty(\Omega_{\delta,r})} \quad \text{and} \quad \|\phi\|_b = \|\phi\|_{L^\infty(\partial\Omega_{\delta,r})},
$$

where $\Omega_{\delta,r} := B_r(0) \setminus \check{B}_{\delta \varepsilon_j}(0)$. Note that $\partial\Omega_{\delta,r} = \cup_{j=1}^m \partial B_{\delta \varepsilon_j}(0) \cup \partial B_r(0)$. By now it is rather standard to show that for functions on $\Omega_{\delta,r}$ the operator $\hat{L}_j$ satisfies the maximum principle in $\Omega_{\delta,r}$ for $\delta$ large and $r > 0$ small enough, see for example [14]. In fact, the function $g(y) := Y_0(a \varepsilon_j^{-1} y) = 2 \frac{a^2 |y|^2 - \mu_j^2 \varepsilon_j^2}{\mu_j^2 \varepsilon_j^2 + |y|^2}$ with $0 < 4a < 1$ and $\sqrt{\frac{9}{3}a} < \delta$ satisfies $\hat{L}_j(y) < 0$ and $g > \frac{1}{2} > 0$ in $\Omega_{\delta,r}$. As a consequence, we get that

**Claim A.5.** There is a constant $C > 0$ such that if $\hat{L}_j(\phi) = h$ in $\Omega_{\delta,r}$ then

$$
\|\phi\|_a \leq C[\|\phi\|_b + \|h\|_\star], \quad (A.7)
$$

where

$$
\|h\|_\star = \sup_{y \in B_r(0)} \frac{|h(y)|}{\rho_{\mu,\varepsilon,m}(y) + 1}.
$$

**Proof:** We shall omit the subscript $a$ in the quantities involved. We will establish this inequality with the use of suitable barriers. Consider now the solution of the problem

$$
-\Delta \psi = 1, \quad \text{for } \delta \varepsilon_j < |y| < r,
$$

$$
\psi(y) = 0 \quad \text{for } |y| = \delta \varepsilon_j, \ |y| = r.
$$

Direct computation shows that

$$
\psi_{j,1}(y) = \frac{\delta^2 \varepsilon_j^2 - r^2}{4} + \frac{\delta^2 \varepsilon_j^2 - r^2 \log |y|}{\log \frac{\delta \varepsilon_j}{r}}.
$$

Note that $0 \leq \psi_{j,1} \leq \frac{\frac{r^2 - \delta^2 \varepsilon_j^2}{4}}{4} \leq \frac{r^2}{4}$ hence these functions $\psi_{j,1}$ have a uniform bound independent of $\varepsilon_j$.

On the other hand, let us consider the function $g$ defined above, and let us set

$$
\psi(y) = 2\|\phi\|_b g(y) + \|h\|_\star[\psi_{j,1}(y) + \psi_{j,2}(y)],
$$

where $\psi_{j,2}$ is the solution to

$$
-\Delta \psi = 2\rho_{\mu,\varepsilon_j,m_j}, \quad \delta \varepsilon_j < |y| < r,
$$

$$
\psi(y) = 0 \text{ for } |y| = \delta \varepsilon_j, \ |y| = r.
$$

Then, it is easily checked that, choosing $\delta$ larger if necessary, $\hat{L}_j(\psi) \leq h$ and $\psi \geq |\phi|$ on $\partial\Omega_{\delta,r}$. Hence $|\phi| \leq \psi$ in $\Omega_{\delta,r}$. In fact, we have that for all $y \in \partial\Omega_{\delta,r}$

$$
\psi(y) \geq 2\|\phi\|_b g(y) \geq \|\phi\|_b \geq |\phi(y)|.
$$

Also, we have that choosing $2C(\delta,r) + r^2 < 1$ (for $\delta$ large enough and $r$ small enough)

$$
\hat{L}_j(\psi) < \|h\|_\star \left( \Delta[\psi_{j,1} + \psi_{j,2}] + \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 \varepsilon_j^2 + |y|^2)^2} [\psi_{j,1} + \psi_{j,2}] \right)
$$

$$
\leq \|h\|_\star \left( -1 - 2\rho_{\mu,\varepsilon_j,m_j}(y) + \frac{8\mu_j^2 \varepsilon_j^2}{(\mu_j^2 \varepsilon_j^2 + |y|^2)^2} [2C(\delta,r) + r^2] \right)
$$

$$
\leq -\|h\|_\star \rho_{\mu,\varepsilon_j,m_j} + 1 \leq h,
$$

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in view of $\rho_{\mu_j, \varepsilon_j, m_j}(y) \geq \frac{8\mu_j^2\varepsilon_j^2}{(\mu_j^2 + |y|^2)^2}$. Hence, we conclude that $|\phi(y)| \leq \psi(y)$ for all $\delta \varepsilon_j < |y| < r$, for every $j = 1, \ldots, k$ and the claim follows.

The following intermediate result provides another estimate. Again, for notational simplicity we omit the subscript $n$ in the quantities involved.

Claim A.6. There exist constants $C > 0$ such that for large $n$

$$\|\phi\|_{L^\infty(B_{r_\alpha}(\xi_j))} \leq C \{\|\phi\|_{L^\infty(B_{r_\alpha}^0(\xi_j))} + o(1)\}. \tag{A.8}$$

Proof: First, note that

$$\|\hat{h}\|_{*} \leq C \left[|c(\hat{\phi})| + \|\hat{h}\|_{*} + \sum_{p=0}^{k} \sum_{q=1}^{k} |c_{pq}| \right]$$

in view of the definition of $\|\cdot\|_{*}$ and $\|\cdot\|_{**}$ and $\|\Delta g P\|_{*} \leq C$. From estimate (A.7) we deduce that there is a constant $C > 0$ such that

$$\|\phi\|_{L^\infty(B_{r}(\xi_j) \setminus B_{s\varepsilon_j}(\xi_j))} = \|\hat{\phi}\|_{a} \leq C \left[|\hat{\phi}|_{b} + \|\hat{h}\|_{**} \right]$$

$$\leq C \left[\|\phi\|_{L^\infty(B_{r}(\xi_j) \setminus B_{s\varepsilon_j}(\xi_j))} + |c(\hat{\phi})| + \|\hat{h}\|_{*} + \sum_{p=0}^{k} \sum_{q=1}^{k} |c_{pq}| \right] \tag{A.9}$$

From (A.6) we find that for large $n$

$$\|\phi\|_{L^\infty(S \setminus \cup_{j=1}^{n} B_{r}(\xi_j))} = o(1). \tag{A.10}$$

Furthermore, we have that $c(\hat{\phi}) = o(1)$, since $c_0 = 0$. By the assumption, we know that $\|\hat{h}\|_{*} = o(1)$. Now, from (A.9) it is clear that

$$\|\phi\|_{L^\infty(B_{r}(\xi_j) \setminus B_{s\varepsilon_j}(\xi_j))} \leq \|\phi\|_{L^\infty(B_{r}(\xi_j) \setminus B_{s\varepsilon_j}(\xi_j))} + C \left[|\hat{\phi}|_{b} + \|\hat{h}\|_{*} \right]$$

and the conclusion follows by (A.10).

We continue with the proof of Proposition 3.1 and we get the following fact.

Claim A.7. There exists an index $j \in \{1, \ldots, k\}$ such that passing to a subsequence if necessary,

$$\liminf_{n \to \infty} \|\phi_n\|_{L^\infty(B_{sz_j}(\xi_j^*))} \geq \kappa > 0. \tag{A.11}$$

Proof: Arguing by contradiction, if for all $j = 1, \ldots, k$

$$\liminf_{n \to \infty} \|\phi_n\|_{L^\infty(B_{sz_j}(\xi_j^*))} = 0,$$

then (A.8) and (A.10) implies that, passing to a subsequence if necessary, $\|\phi_n\|_{\infty} \to 0$ as $n \to +\infty$. On the other hand, we know that $\|\phi\|_{\infty} = 1$ for all $n \in \mathbb{N}$. This conclude (A.11).

Let us set $\psi_{n,j}(z) = \phi_n(y_{e_j}(\varepsilon_j^* z)) = \hat{\phi}_{n,j}(\varepsilon_j^* z)$ for any $j$ and $z \in B_{r_\alpha}(0)$. We notice that $\psi_{n,j}$ satisfies

$$\Delta \psi_{n,j} + \frac{8|\mu_j|^2}{(\mu_j^2 + |z|^2)^2} \psi_{n,j} = \varepsilon_j^2 \hat{h}_{n,j}(\varepsilon_j^* z) \text{ in } B_{r_\alpha}(0).$$

Elliptic estimates and (A.11) readily imply that $\psi_{n,j}$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, non-zero solution $\psi_j^*$ of

$$\Delta \psi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2} \psi = 0, \quad \mu_j = \mu_j^*$$
in view of \( \|\varepsilon^n\|^2 h_{n,j}(\varepsilon^n z) \leq C\|h_n\| \) for \( z \) in compact subsets of \( \mathbb{R}^2 \). This implies that \( \psi^*_i \) is a linear combination of the functions \( Y_{ij}, i = 0, 1, 2 \). Thus, we have that for some constants \( a_{ij}, i = 0, 1, 2, \psi^*_i = a_{0j}Y_{0j} + a_{1j}Y_{1j} + a_{2j}Y_{2j} \). See [5] for a proof. But, from (A.2), orthogonality conditions over \( \psi_{n,j} \) pass to the limit thanks to \( \|\psi_{n,j}\|_\infty \leq C \) and dominated convergence, namely,

\[
\int_{\mathbb{R}^2} \Delta Y_{ij} \psi^*_j = 0, \quad \text{for } i = 0, 1, 2.
\]

A contradiction with (A.11) arises since this implies that \( a_{0j} = a_{1j} = a_{2j} = 0 \). Thus, we get the estimate \( \|\phi\|_\infty \leq C\|h\|_\ast \). Hence, from the same argument shown in the proof of the Claim (A.2) we conclude the estimates (3.4).

Now, let us prove the solvability assertion. To this purpose we consider the space

\[
H = \left\{ \phi \in H^1_0(S) : \int_S \Delta_\psi PZ_{ij} \phi = 0 \quad \text{for } i = 0, 1, 2, j = 1, \ldots, k \right\},
\]

endowed with the usual inner product \( \langle \phi, \psi \rangle = \int_S (\nabla \phi, \nabla \psi) \, dv_g \). Problem (3.3) expressed in weak form is equivalent to that of finding a \( \phi \in H \), such that

\[
\langle \phi, \psi \rangle = \int_S [K\phi - \hat{h} \psi] \, dv_g, \quad \text{for all } \psi \in H.
\]

Recall that \( \int_S h \, dv_g = 0 \). With the aid of Riesz’s representation theorem, this equation gets rewritten in \( H \) in the operator form \( \phi = K(\phi) + \hat{h} \), for certain \( \hat{h} \in H \), where \( K \) is a compact operator in \( H \). Fredholm’s alternative guarantees unique solvability of this problem for any \( h \) provided that the homogeneous equation \( \phi = K(\phi) \) has only the zero solution in \( H \). This last equation is equivalent to (3.3) with \( h \equiv 0 \). Thus existence of a unique solution follows from the a priori estimate (3.4).

We have just proven that the unique solution \( \phi = T_\lambda(h) \) of (3.3) defines a continuous linear map from the Banach space \( C \ast \) of all functions \( h \in L^\infty(S) \) for which \( \|h\|_\ast < +\infty \) and \( \int_S h = 0 \), into \( L^\infty \), with bounded norm.

It is important to understand the differentiability of the operator \( T \) with respect to the variable either \( \beta = \xi_i \) or \( \beta = m_j \). Fix \( h \in C \ast \) and let \( \phi = T_\lambda(h) \). Let us recall that \( \phi \) satisfies (3.3), for some (uniquely determined) constants \( c_{ij}, i = 0, 1, 2, j = 1, \ldots, k \). We want to compute derivatives of \( \phi \) with respect to the parameters \( \beta = \xi_i \) or \( \beta = m_j \). Formally \( X = \partial_\beta \phi \) should satisfy

\[
L(X) = -\partial_\beta K \phi + \frac{1}{|S|} \int_S \partial_\beta K \phi + \sum_{i=0}^2 \sum_{j=1}^k c_{ij} \partial_\beta (\Delta_\psi PZ_{ij}) + \sum_{i=0}^2 \sum_{j=1}^k d_{ij} \Delta_\psi PZ_{ij},
\]

where (still formally) \( d_{ij} = \partial_\beta (c_{ij}) \), \( i = 0, 1, 2, j = 1, \ldots, k \). The orthogonality conditions now become

\[
\int_S \Delta_\psi PZ_{ij} X = -\int_S \partial_\beta (\Delta_\psi PZ_{ij}) \phi, \quad i = 0, 1, 2, j = 1, \ldots, k.
\]

We will recast \( X \) as follows. Consider the \( Y = X + \sum_{i,j} b_{ij} \psi_{ij} \), where the coefficients \( b_{ij} \) are chosen so that \( Y \) satisfies the orthogonality conditions \( \int_S Y \Delta_\psi PZ_{ij} = 0 \) for all \( i, j \). The coefficients \( b_{ij} \) are well-defined since they satisfy an almost diagonal system in view of (A.4).
Furthermore, it holds that for \( \beta = \xi_l \), \( |b_{ij}| \leq C\|h\|_* \) if \( j \neq l \) and \( |b_{il}| \leq \frac{C}{\xi_l} \|h\|_* \); and for \( \beta = m_l \), \( |b_{ij}| \leq C\|h\|_* \) if \( j \neq l \) and \( |b_{il}| \leq C|\log \varepsilon_l|\|h\|_* \). In view of \( |\partial_{\xi_l}(\Delta_g P_{Zij})|_* \leq C \) if \( j \neq l \) and \( |\partial_{m_l}(\Delta_g P_{Zij})|_* \leq C|\log \varepsilon_l| \). Then the function \( X \) above can be uniquely expressed as

\[
X = T_\lambda(f) - \sum_{i=0}^{2} \sum_{j=1}^{k} b_{ij} P_{Zij}.
\]

where the function

\[
f := -\partial_{\beta} K \phi + \frac{1}{|S|} \int_S \partial_{\beta} K \phi + \sum_{i=0}^{2} \sum_{j=1}^{k} [b_{ij} L(PZ_{ij}) + c_{ij} \partial_{\beta}(\Delta_g P_{Zij})].
\]

This computation is not just formal. Arguing directly by definition it shows that indeed \( \partial_{\beta} \phi = X \). Also, we find that \( \|f\|_* \leq \frac{C}{\varepsilon_l} \|h\|_* \), for \( \beta = \xi_l \) and \( \|f\|_* \leq C|\log \varepsilon_l|\|h\|_* \), for \( \beta = m_l \), in view of

\[
\|f\|_* \leq \|\partial_{\beta} K \phi\|_* \|\phi\|_{\infty} \left[ 1 + \frac{1}{|S|} \int_S \rho(x) dv_g \right] + \sum_{i=0}^{2} \sum_{j=1}^{k} [b_{ij} \|L(PZ_{ij})\|_* + c_{ij}|\partial_{\beta}(\Delta_g P_{Zij})|_*].
\]

Indeed, it is easy to check that \( \int_S \rho(x) dv_g \leq C \), \( \|L(PZ_{ij})\|_* \leq C \varepsilon_j \) for \( i = 1, 2 \) and \( \|L(PZ_{ij})\|_* \leq C \). Furthermore, from the definition of \( K \) it follows that \( \|\partial_{\xi_l} K\|_* \leq \frac{C}{\varepsilon_l} \) and \( \|\partial_{m_l} K\|_* \leq C|\log \varepsilon_l| \). Moreover, using estimate (3.4) applied with R.H.S. \( f \), we find that

\[
\|\partial_{\xi_l} \phi\|_{\infty} \leq C \|f\|_* + \sum_{i=0}^{2} \sum_{j=1}^{k} |b_{ij}|,
\]

so that, \( \|\partial_{\beta} \phi\|_* \leq \frac{C}{\varepsilon_l} \|h\|_* \) and \( \|\partial_{m_l} \phi\|_{\infty} \leq C|\log \varepsilon_l|\|h\|_* \). Finally, we conclude that

\[
\|\partial_{(\xi_l)_l} T_\lambda(h)\|_{\infty} \leq \frac{C}{\varepsilon_l} \|h\|_* \quad \text{for} \quad i = 1, 2, l = 1, \ldots, k \quad \text{(A.12)}
\]

and

\[
\|\partial_{m_l} T_\lambda(h)\|_{\infty} \leq C|\log \varepsilon_l|\|h\|_* \quad \text{for} \quad l = 1, \ldots, k. \quad \text{(A.13)}
\]

This finishes the proof of proposition 3.1.

**Appendix B: The nonlinear problem**

By Proposition 3.1 we now deduce the following.

**Proof (of Proposition 3.2).** First, note that \( R \in L^\infty(S), \|R\|_* < +\infty, \int_S R = 0 \) and \( \int_S N(\phi) = 0 \) for any \( \phi \in C(S) \). Next, we observe that in terms of the operator \( T \) defined in Proposition 3.1, the latter problem becomes

\[
\phi = -T (R + N(\phi)) := \mathcal{A}(\phi).
\]
For a given number \( \nu > 0 \), let us consider
\[
F_\nu = \{ \phi \in C(S) : \| \phi \|_\infty \leq \nu \lambda \}
\]
From the Proposition 3.1, we get
\[
\| A(\phi) \|_\infty \leq C \| R \|_* + \| N(\phi) \|_*.
\]
From \([2.13]\) we know that \( \| R \|_* \leq CA \). Furthermore, it follows that for certain \( 0 < s < s^* < 1 \)
\[
f(V_+\phi) - f(V) - f'(V)\phi = \int_0^1 [f'(V+t\phi) - f'(V)] dt = [f'(V+s\phi) - f'(V)]\phi = f''(V+s\phi) s\phi^2.
\]
From the definition of \( f \) in \([2.23]\) and the estimates used to prove Lemma \([2.2]\) it follows that
\[
f''(V + s\phi) = 2\lambda^2 (V + s\phi)e^{\lambda(V+s\phi)^2} [3 + 2\lambda(V + \phi)^2] = \lambda V e^{\lambda V^2} O(1) + \lambda^2 e^{\lambda V^2} O(1)
\]
so that, from the definition of \( \| \cdot \|_* \), we obtain that \( \| f''(V + s\phi) \|_* \leq C \). Thus, we find that
\[
\| N(\phi) \|_* \leq \left[ \| f(V + \phi) - f(V) - f'(V)\phi \|_* + \| f'(V) - K \| \| \phi \|_\infty \right] 1 + \frac{1}{\|s\|} \int_S \rho(x) dv_g
\]
\[
\leq C \| \phi \|_\infty^2 + \lambda \| \phi \|_\infty \leq C\lambda^2 [\nu^2 + \nu],
\]
in view of \([2.26]\) and \( \int_S \rho(x) dv_g \leq C \). Hence, we get for any \( \phi \in F_\nu \),
\[
\| A(\phi) \|_\infty \leq C\lambda \left[ 1 + (\nu + \nu^2)\lambda \right].
\]
On the other hand, for \( \phi_1 \) and \( \phi_2 \) and certain \( 0 < s, t^* < 1 \) we have that
\[
f(V + \phi_1) - f(V + \phi_2) - f'(V)(\phi_1 - \phi_2)
\]
\[
= \int_0^1 [f'(V + \phi_1 + t\phi_1 - \phi_2)] - f'(V)] dt [\phi_1 - \phi_2]
\]
\[
= [f'(V + t^*\phi_1 + (1-t^*)\phi_2) - f'(V)] [\phi_1 - \phi_2]
\]
\[
= f''(V + s[t^*\phi_1 + (1-t^*)\phi_2)] [t^*\phi_1 + (1-t^*)\phi_2] [\phi_1 - \phi_2],
\]
so that,
\[
\| f(V + \phi_1) - f(V + \phi_2) - f'(V)(\phi_1 - \phi_2) \|_* \leq C \| \phi_1 \|_\infty + \| \phi_2 \|_\infty \| \phi_1 - \phi_2 \|_\infty,
\]
in view of \( \| f''(V + s[t^*\phi_1 + (1-t^*)\phi_2]) \|_* \leq C \). Hence, given any \( \phi_1, \phi_2 \in F_\nu \), we have that
\[
\| N(\phi_1) - N(\phi_2) \|_* \leq C(\| \phi_1 \|_\infty + \| \phi_2 \|_\infty)\| \phi_1 - \phi_2 \|_\infty + C\lambda \| \phi_1 - \phi_2 \|_\infty
\]
\[
\leq C\lambda [\nu + 1] \| \phi_1 - \phi_2 \|_\infty
\]
with \( C \) independent of \( \nu \). Therefore, from the Proposition 3.1
\[
\| A(\phi_1) - A(\phi_2) \|_\infty \leq C\| N(\phi_1) - N(\phi_2) \|_* \leq C\lambda [\nu + 1] \| \phi_1 - \phi_2 \|_\infty.
\]
It follows that for all \( \varepsilon \) sufficiently small \( A \) is a contraction mapping of \( F_\nu \) (for \( \nu \) large enough), and therefore a unique fixed point of \( A \) exists in \( F_\nu \).

Let us now discuss the differentiability of \( \phi \) depending on \((\xi, m)\), i.e., \( (\xi, m) \mapsto \phi(\xi, m) \in C(S) \) is \( C^1 \). Since \( R \) depends continuously (in the *-norm) on \((\xi, m)\), using the fixed point
characterization (15.1), we deduce that the mapping \((\xi, m) \mapsto \phi\) is also continuous. Then, formally
\[
\partial_\beta N(\phi) = [f'(V + \phi) - f'(V) - f''(V)\phi] \partial_\beta V + [f'(V + \phi) - f'(V)] \partial_\beta \phi
\]
\[
- \frac{1}{|S|} \int_S \left\{ [f'(V + \phi) - f'(V) - f''(V)\phi] \partial_\beta V + [f'(V + \phi) - f'(V)] \partial_\beta \phi \right\}
\]
\[
+ [f''(V)\partial_\beta V - \partial_\beta K]\phi + [f'(V) - K]\partial_\beta \phi
\]
\[
- \frac{1}{|S|} \int_S ([f''(V)\partial_\beta V - \partial_\beta K]\phi + [f'(V) - K]\partial_\beta \phi).
\]
so that, we estimate as follows
\[
\|\partial_\beta N(\phi)\|_* \leq C \left[ \|f''(V + s'\phi) - f''(V)\|_* \|\phi\|_\infty \|\partial_\beta V\|_\infty + \|f''(V + s'\phi)\|_* \|\phi\|_\infty \|\partial_\beta \phi\|_\infty \right.
\]
\[
+ \|f''(V)\partial_\beta V - \partial_\beta K\|_* \|\phi\|_\infty + \|f'(V) - K\|_* \|\partial_\beta \phi\|_\infty \right],
\]
for some \(s' \in (0, 1)\). In particular, precisely for \(\beta = \xi_l\) we obtain
\[
\|\partial_{\xi_l} N(\phi)\|_* \leq C \left[ \|\phi\|_{L^\infty}^2 \|\partial_{\xi_l} V\|_\infty + \|\phi\|_\infty \|\partial_{\xi_l} \phi\|_\infty + \frac{\lambda}{\xi_l} \|\phi\|_\infty + \lambda \|\partial_{\xi_l} \phi\|_\infty \right]
\]
\[
\leq C \left[ \frac{\lambda^2}{\xi_l} + \lambda \|\partial_{\xi_l} \phi\|_\infty \right],
\]
in view of \(\|f''(V + s'\phi) - f''(V)\|_* \leq C \|\phi\|_\infty, \|\partial_{\xi_l} V\|_\infty \leq C \|\xi_l\|, \|\partial_{\xi_l} \phi\|_\infty \leq \frac{C\lambda}{\xi_l}\) and estimate (2.26), and for \(\beta = m_l\) we obtain
\[
\|\partial_{m_l} N(\phi)\|_* \leq C \left[ \|\phi\|_{L^\infty}^2 \|\partial_{m_l} V\|_\infty + \|\phi\|_\infty \|\partial_{m_l} \phi\|_\infty + \lambda |\log \varepsilon_l| \|\phi\|_\infty + \lambda \|\partial_{m_l} \phi\|_\infty \right]
\]
\[
\leq C \left[ \lambda^2 |\log \varepsilon_l| + \lambda \|\partial_{m_l} \phi\|_\infty \right],
\]
in view of \(\|\partial_{m_l} V\|_\infty \leq C |\log \varepsilon_l|, \|f''(V)\partial_{m_l} V - \partial_{m_l} K\|_* \leq C\lambda |\log \varepsilon_l| \leq C\). Also, observe that we have
\[
\partial_\beta \phi = - (\partial_\beta T) (R + N(\phi)) - T (\partial_\beta [R + N(\phi)]).
\]
So, using (A.12) and previous estimates, we get
\[
\|\partial_{\xi_l} \phi\|_\infty \leq \frac{C}{\varepsilon_l} \|R + N(\phi)\|_* + C \|\partial_{\xi_l} (R + N(\phi))\|_*
\]
\[
\leq \frac{C}{\varepsilon_l} \left[ \lambda + \lambda \|\phi\|_\infty + \|\phi\|_{L^\infty}^2 \right] + C \left[ \frac{\lambda}{\varepsilon_l} + \frac{\lambda^2}{\varepsilon_l} + \lambda \|\partial_{\xi_l} \phi\|_\infty \right]
\]
\[
\leq \frac{C\lambda}{\varepsilon_l} + C\lambda \|\partial_{\xi_l} \phi\|_\infty.
\]
and similarly, using (A.13) and previous estimates, we get
\[
\|\partial_{m_l} \phi\|_\infty \leq C |\log \varepsilon_l| \|R + N(\phi)\|_* + C \|\partial_{m_l} (R + N(\phi))\|_*
\]
\[
\leq C |\log \varepsilon_l| \left[ \lambda + \lambda \|\phi\|_\infty + \|\phi\|_{L^\infty}^2 \right] + C \left[ \lambda |\log \varepsilon_l| + \lambda^2 |\log \varepsilon_l| + \lambda \|\partial_{\xi_l} \phi\|_\infty \right]
\]
\[
\leq C\lambda |\log \varepsilon_l| + C\lambda \|\partial_{\xi_l} \phi\|_\infty.
\]
We have an estimate for \(\|\partial_\beta R\|_*\). From the definition of \(V\), the definition of \(\partial_\beta R\)
\[
\partial_\beta R(y) = \Delta_y \partial_\beta V(y) + f'(V)\partial_\beta V - \frac{1}{|S|} \int_S f'(V)\partial_\beta V \, dv_y,
\]
similar computations to deduce (2.13) and from the definition of $\ast$-norm it follows that

$$\| \partial_{\xi_l} R \|_\ast \leq \frac{C\lambda}{\varepsilon_t} \quad \text{and} \quad \| \partial_{m_l} R \|_\ast \leq C\lambda \log \varepsilon_t.$$  

Thus, we conclude (3.8).

The above computations can be made rigorous by using the implicit function theorem and the fixed point representation (3.11) which guarantees $C^1$ regularity in $(\xi, m)$. \qed

**Appendix C: Proof of Lemma 3.3**

**Proof:** Let us differentiate the function $F_{\lambda}(\xi, m)$ with respect to either $\beta = (\xi_i)_q$ or $\beta = m_l$, with $q = 1, 2$ and $l = 1, \ldots, k$. Since $\sqrt{X}V(\xi, m) = U(\xi, m)$ and $\sqrt{X}\phi(\xi, m) = \phi(\xi, m)$, we can differentiate directly $J_{\lambda} (\sqrt{X[V + \phi]})$ (under the integral sign), so that integrating by parts we get

$$
\partial_{\beta} F_{\lambda}(\xi, m) = \sqrt{X}D_{J_{\lambda}} \left( \sqrt{X}[V + \phi] \right) \partial_{\beta} V + \partial_{\beta} \phi
\quad = -\lambda \int_S \left[ \Delta_g(V + \phi) + \lambda(V + \phi)e^{X(V + \phi)} \right] \partial_{\beta} V + \partial_{\beta} \phi \, dv_g
\quad = -\lambda \sum_{i=0}^2 \sum_{j=1}^k c_{ij} \int_S \Delta_gDZ_{ij} \left[ \partial_{\beta} V + \partial_{\beta} \phi \right],
$$

since $\int_S (\partial_{\beta} V + \partial_{\beta} \phi) = 0$. From the result of 3.2, this expression defines a continuous function of $(\xi, m)$. Let us stress that $\mu_j = \mu_j(\xi, m)$ and $\varepsilon_j = \varepsilon_j(m_j)$. Hence, from 4.7 we have that uniformly in $S$

$$
\partial_{(\xi_i)_q} V(x) = -\frac{m_l}{\mu_l} \chi_l \frac{2\mu_l \varepsilon_l \partial_{(\xi_i)_q} (|y_{\xi_l}(x)|^2)}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(1)
$$

and from 4.8

$$
\partial_{m_l} V(x) = -2\chi_l \log(\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2) + \chi_l \frac{8\mu_l^2 \varepsilon_l^2 \log \varepsilon_l}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(1),
$$

in view of $\varepsilon_l \partial_{m_l} \varepsilon_l = -\frac{2\varepsilon_l^2}{m_l} \log \varepsilon_l + O(1)$. Let us assume that $D_{\xi}F_{\lambda}(\xi, m) = 0$ and $D_{m}F_{\lambda}(\xi, m) = 0$.

Then, from the latter equality and the estimates (3.8) we get

$$
\sum_{i=0}^2 \sum_{j=1}^k c_{ij} \int_S \Delta_gPZ_{ij} \left[ \varepsilon_l \partial_{(\xi_j)_q} V + O(\lambda) \right] = 0, \quad q = 0, 1, 2, \ l = 1, \ldots, k
$$

and also,

$$
\sum_{i=0}^2 \sum_{j=1}^k c_{ij} \int_S \Delta_gPZ_{ij} \left[ \frac{\partial_{m_l} V}{\log \varepsilon_l} + O(\lambda) \right] = 0, \quad l = 1, \ldots, k.
$$

Using

$$
\varepsilon_l \partial_{(\xi_j)_q} V = -\frac{m_l}{\mu_l} \chi_l \frac{2\mu_l \varepsilon_l \partial_{(\xi_j)_q} (|y_{\xi_l}(x)|^2)}{\mu_l^2 \varepsilon_l^2 + |y_{\xi_l}(x)|^2} + O(\varepsilon_l)
$$

and

$$
\frac{\partial_{m_l} V}{\log \varepsilon_l} = \frac{1}{\log \varepsilon_l} \chi_l [U_l - \log(8\mu_l^2 \varepsilon_l^2)] + 2\chi_l |Z_{ul}| + O\left(\frac{1}{\log \varepsilon_l} \right),
$$

39
where $O(\varepsilon_l)$ and $O(|\log \varepsilon_l|^{-1})$ are in the $L^\infty$ norm as $\lambda \to 0$, it follows

\[
2 \sum_{i=0}^2 \sum_{j=1}^m c_{ij} \int_S \Delta g P_{Zij} \left[ \chi_l \left( 2 \frac{\mu \varepsilon_l}{\mu_l^2 \varepsilon_l^2} + 2 \right) + o(1) \right] = 0, \quad q = 0, 1, 2, \ l = 1, \ldots, m.
\]

\[
2 \sum_{i=0}^2 \sum_{j=1}^m c_{ij} \int_S \Delta g P_{Zij} \left[ \chi_l \left( Z_{0l} + 2 \right) + o(1) \right] = 0, \quad l = 1, \ldots, m.
\]

with $o(1)$ small in the sense of the $L^\infty$ norm as $\lambda \to 0$. The above system is diagonal dominant and we thus get $c_{ij} = 0$ for $i = 0, 1, 2, j = 1, \ldots, k$. We have used that

\[
\int_S \Delta g P_{Zij} \chi_l [U_l - \log(8\mu_l^2 \varepsilon_l^2)] = O(1).
\]

The proof of Lemma 3.3 is finished. \(\square\)

References


