PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
SCHOOL OF ENGINEERING

# OPTIMAL IPO TIMING: A GENERAL EQUILIBRIUM APPROACH IN ENDOWMENT ECONOMIES

**MAURO ALFONSO VILLALÓN SEPÚLVEDA**

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
JAIME CASASSUS V.

Santiago de Chile, January 2011

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*A Tanya y Amalia, mis mayores incentivos a siempre seguir adelante*

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## ABSTRACT

We model the IPO decision of an entrepreneur in an exchange economy and study its diversification benefits. The entrepreneur holds an unlisted asset or firm, modelled as a Lucas Tree, and when the IPO occurs the market converges to a Two Trees economy built on Cochrane, Longstaff, & Santa-Clara (2007). We solve the optimal timing problem and study the diversification effects over the asset's value and entrepreneur's consumption. Both, entrepreneur and market benefit from the IPO and the new diversified scenario, but because the decision involves just one of the agents, the optimal IPO timing does not maximise the economy's aggregate utility. Taxes and subsidies solve this problem. The model predicts that IPOs should be correlated with the firm's size and explains why firms with lower betas are expected to IPO first.



**Keywords:** Optimal IPO Timing, General Equilibrium, Developing Economies.

**RESUMEN**

Modelamos la decisión de IPO de un empresario en una economía de intercambio y estudiamos los beneficios de diversificación. El empresario inicialmente posee un activo no listado en la bolsa, modelado como árbol de Lucas, y en el momento del IPO el mercado converge a una economía de dos árboles, desarrollada en Cochrane, Longstaff, & Santa-Clara (2007). Resolvemos el problema de tiempo óptimo, y estudiamos los efectos de diversificación sobre el valor de la firma y el consumo del empresario. Tanto el empresario como el mercado se benefician por el IPO y el nuevo escenario diversificado, pero ya que la decisión es tomada por un solo agente, el tiempo óptimo de IPO no maximiza el bienestar de la economía agregada. Esto puede solucionarse con la implementación de subsidios e impuestos. El modelo predice que los IPO debieran estar correlacionados con el tamaño de las firmas dentro de una industria, y explica por qué se espera que las primeras firmas en abrirse al mercado tengan menores betas.

**Palabras Claves:** Tiempo Óptimo de IPO, Equilibrio General, Economías en Desarrollo.

## 1. INTRODUCTION

Market conditions has been considered, by most empirical studies, as one of the most important factors that could trigger an Initial Public Offerings (IPO) decision. Empirical evidence, such as the so called ”IPO Waves” (Pástor & Veronesi, 2005; Ritter, 1984), suggests that entrepreneurs wait for certain conditions on the market to IPO their companies. IPO waves are also characterized by the presence of firms from similar sectors overweighting particular industries in the market portfolio. IPO decision should not only be determined by market conditions, but also by characteristics shared within firms from similar industries.

IPO studies have mostly been focused in developed and mature economies. These economies are generally well balanced, and comparatively big in relation to a newly listed firm implying that the introduction of the new firm has no impact in its industrial composition. However, this is not true for developing economies, where markets are immature and the industrial composition of the economy is not well represented in the market portfolio. In emerging markets, large firms or sectors that IPO may deeply impact the composition and characteristics of the market portfolio, empirically documented in Braun & Larrain (2009). The IPO decision is inevitably influenced by its effect on the overall economy.

Benninga, Helmantel, & Sarig (2005) summarizes several motives that might encourage an entrepreneur to IPO: financial support for investment opportunities, a possible increase in the company’s valuation due to factors such as external monitoring and liquidity of shares (see also Holmström & Tirole (1993)), and differences in the way owners and buyers price the company (undiversified versus diversified entrepreneurs respectively). Nevertheless, these factors have some problems in determining a generalized theory of initial public offerings. Financial support can be obtained through private equity and bank loans, and it has been documented that investments in firms actually decline after an IPO (e.g. Pagano, Panetta, & Zingales (1998) and Benninga, Helmantel, & Sarig (2005)). Factors such as external monitoring may increase firm’s value, but as shown in Datta, Iskandar-Datta, & Patel

(1999), some monitoring costs can be lowered using bank debt (which at the same time reduces public debt borrowing costs). Differences in valuation due to entrepreneurs diversification may be diminished by credibility issues due to information asymmetries (Courteau (1995)).

Many authors argue that diversification may play a central role in IPOs. Bodnaruk, Kandel, Massa, & Simonov (2008) finds empirical evidence that relates portfolio diversification with IPOs: *Less diversified shareholders are more eager to take their company public*. This effect is more important when individuals are more risk-averse. Courteau (1995) argues that after the IPO the owner can diversify some firm-specific risk (non systematic risk) and get a better balanced investment portfolio. Astudillo (2008) study the empirical process in which markets mature. In immature markets, few industries of the economy are well represented, and evidence indicates that low beta sectors are overweighted in the market composition. This means that the first companies expected to IPO should have lower market betas, which lowers the market systematic risk.

Benninga et al. (2005) clearly states that IPOs are not single-shot opportunities: entrepreneurs have the ability to time their IPOs. If we suppose IPO decisions are irreversible (an entrepreneur is not planning to reacquire the company, at least not soon enough to consider it in the decision), real option analysis seems like a natural candidate to model this kind of problems. Literature regarding the subject has not agreed in a conclusive model, but several authors had already been attempting real option approaches with different conclusions. Pástor & Veronesi (2005) consider the IPO as an option, analysing the trade-off between the benefits of waiting for a better discount rate and the abnormal returns from new technologies. Draho & Stanley (2000) elaborates a real option model where the firm waits for high prices in public companies from the same industry. Benninga et al. (2005) uses a binomial option-pricing model, in which diversified investors are willing to pay higher prices for the risky cash flows. The counterpart of the IPO loses private benefits of control. Pastor, Taylor, & Veronesi (2009) develops an optimal timing model, where the benefits of private control are counterweighted by diversification benefits of an IPO. Chen, Miao, & Wang (2009) has recently develop a partial equilibrium model in which an entrepreneur maximises his utility. When starting a business, the entrepreneur may choose to diversify his consumption either financing business with riskless debt (in order to maintain a higher liquid diversificated capital), through risky debt or cashing out his business. In worse scenarios, the entrepreneur may always default on his debt, liquidating his firm.

In all of these partial equilibrium models, the entrepreneur considers the trade-off between waiting for better conditions in business and market in order to IPO (or sell his firm) versus costs associated to the action. The IPO decision is influenced by the market, but the market stays untouched by the new listed company. These models may be applied to economies in which trading markets are comparatively big and mature, but not to developing and emerging economies, in which an IPO may change market conditions, diversification scenarios, its industrial compositions, risks and expected returns. A new company or group of companies going public may affect the market composition and both, entrepreneur and market knows this before the asset is publicly listed.

We propose a simple model in an exchange endowment economy, in which the benefits from an IPO comes purely from diversification, eliminating any other incentive that might trigger an IPO (as the ones mentioned above). The whole economy will consist in two risky assets, the entrepreneur's firm and the public market. The entrepreneur does not have a significant liquid capital to invest in a diversified portfolio, and his only source of financing comes from equity. We prefer to discard in first instance the presence of debt instruments, in order to simplify the capital structure of the firm. The unlisted asset is represented as an asset paying continues dividend stream that has some firm-specific characteristics. Why modelling in an exchange endowment economy? As is well known, an exchange economy has as a main advantage its simplicity. We can obtain nice and intuitive solutions without caring on how goods and services are produced. This helps us to concentrate exclusively in the diversification benefits from trading perceived by the entrepreneur.

We also wish to maintain transactional costs simple: A constant fraction of the overall wealth of the economy. This supposition is mainly based in two ideas: (i) Empirical facts documented by Ritter (1987), in which he estimates the existence of fixed and variable costs, that are a constant fraction of the economy and IPO gross proceeds respectively[1], and (ii) The lose of private benefits control (see for example Benninga et al. (2005) and Pastor et al. (2009)) should also be proportional to the asset's value or IPO gross proceeds.

In our model, the IPO decision depends on the dividend of the unlisted asset relative to the aggregate consumption in the economy, i.e. the *dividend ratio*. Before the IPO, the entrepreneur agent prefers to keep his asset private because the market value of his asset minus transactional costs does not compensate diversification benefits. Under this scenario, in order to IPO, the incomes from the operation should be larger than the actual value of the asset measured from a diversified investor. At the IPO time the option value is zero and the entrepreneur is willing to sell his company at its market value minus the transactional costs associated. The diversification effect can be measured through the entrepreneur's consumption stream. The dividend ratio required by the entrepreneur to IPO his firm is lower when the potential benefits from diversification effect are higher. Because of the newly diversified market scenario, both agents, entrepreneur and market are benefited with an IPO. Nevertheless, the decision is based in the maximisation of the entrepreneur's private welfare, which does not represent the aggregate economy's optimal IPO timing. As shown in this work, we may tax the market agent and subsidy the entrepreneur so that the IPO occurs at a level that is optimal for the aggregate economy.

In line with recent empirical findings, our model predicts that firms with lower expected betas at the IPO time are expected to IPO first. Our model predicts that IPOs from the same industry (similar dividend streams) should be positively correlated with the company's size, in line with Chemmanur, He, & Nandy (2009) and Pagano, Panetta, & Zingales (1998). We also allow for possible discontinuities in the dividend streams possibly due to economic disasters or technological changes.

This thesis is related to a recent literature that studies the asset pricing implications in exchange economies. Cochrane et al. (2007) builds on Lucas (1978) and study the effect of having a second Lucas trees in the economy. Martin (2007) generalizes the utility

[1]Ritter (1987) estimates, using information from IPOs in the U.S. in 1977-82, that direct costs of going public have a fixed part (250.000 $USD) plus a variable component (7% of the gross proceeds.)

function, dividend processes, and number of trees in the economy (a Lucas orchard). Other asset-pricing studies that consider multiple trees are Longstaff (2009), Parlour, Stanton, & Walden (n.d.) and Parlour, Stanton, & Walden (2009), among others.

The thesis is structured as follows. Chapter 2 develops the framework in which the model is constructed, and solves the optimal control problem associated with the IPO. Chapter 3 summarizes some asset pricing results that are used in the resolution of the model and in the development of the analysis, which may be skipped in a first reading. In Chapter 4, we analyse some results derived from the model, studying sensibility of optimal states to parameters, and after versus before IPO scenarios. Finally Chapter 5 summarizes and concludes this thesis.

## 2. THE MODEL

This chapter develops the model and its assumptions. The chapter is structured as follows. First we define the assets and agents that will be interacting in the economy. Then we state the conditions imposed over the entrepreneur when the IPO is held. Finally we set the optimal control optimization problem, which solution is summarized in Proposition (2.1)

### 2.1. Basic Assumptions

First of all, we will begin defining the representative assets involved in the economy. We wish to represent the assets as Lucas trees with positive dividends, each of them following Lévy jump diffusion processes. This way we may be able to model occasional disasters and technological positive shocks, in addition to the diffusion process.

In the economy we will suppose two representative agents. Each of their preferences will be modeled with a Constant Relative Risk Aversion (CRRA) utility function, discounted by a constant time preference rate (patience factor). This way we may be able to study how risk aversion affects entrepreneur's IPO decisions.

#### 2.1.1. The Financial Market

Let $Z_t$ be a 2-dimensional brownian motion defined in the filtered probability space $(\Omega, \mathcal{F}^Z, (\mathcal{F}_t^Z)_t, \mathbb{P})$. Let $N$ be an 1-dimensional Poisson random measure defined in $(\Omega, \mathcal{F}^N, (\mathcal{F}_t^N)_t, \mathbb{P})$. We define the general filtered probability space as $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$=$\big(\Omega, \mathcal{F}^Z \otimes \mathcal{F}^N, (\mathcal{F}^Z \otimes \mathcal{F}_t^N)_t, \mathbb{P}\big)$, which is the one we are going to work with. For simplifying notation, we designate the operator $\mathbb{E}_t\left[*\right] \equiv \mathbb{E}\left[* \setminus \mathcal{F}_t\right]$.

Suppose an exchange economy with two assets, $P_1$ for an asset that is going public at some time in the future, and $P_2$ for an asset representing the tradable market. Each of the assets pay dividend streams $D_{1t}$ and $D_{2t}$ respectively. We model the dividends as

$D_{it} = e^{y_{it}}$, where $y_t = (y_{1t}, y_{2t})^T$ is defined by the process

$$y_t = y_0 + \mu t + \sigma Z_t + \int_{\mathbb{R}} g(z)\, N(dt, dz) \tag{2.1}$$

Where $g(z)$ is an $2 \times l$ matrix function. $\mu$ is a 2-dimensional constant and $\sigma$ an $2 \times 2$ matrix. We will consider $Z_{1t}$ as the idiosyncratic diffusion component of $P_1$ and $Z_{2t}$ as the systematic market diffusion component. Under this suppositions, $\sigma_{21} = 0$. Using Ito's differential formula for jump diffusion we can write the dividend processes as

$$\frac{dD_{it}}{D_{it}} = \mu_{D_i} dt + \sigma_i dZ_t + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{g_{ik}(z)} - 1)\, N^{(k)}(dt, dz) \tag{2.2}$$

Where $\mu_{D_i} = \left[\mu_i + \frac{1}{2}\sigma_i \sigma_i^T\right]$, with $\sigma_i$ the $i^t h$ row of $\sigma$.

We define the state process $s : \Omega \times \mathbb{R} \to [0, 1]$ as the contribution of the company's dividends to the aggregate consumption, i.e. $s_t = \frac{D_{1t}}{D_{1t}+D_{2t}}$. In some cases it will be easier to work with a monotonic transformation of $s_t$, given by $u_t = \log\left(\frac{1-s_t}{s_t}\right)$.

### 2.1.2. Entrepeneur and Market

There exists two representative agents in the economy: $E$ for the entrepreneur that owns the unlisted asset and $M$ for the rest of the world. Each of the agents has an utility function in $t$ (in terms of consumption $C$) given by

$$U_t(C) = \mathbb{E}_t \left[\int_t^{\infty} e^{-\delta(s-t)} u(C_s)\, ds\right] \tag{2.3}$$

with

$$u(C_t) = \frac{C_t^{1-\gamma} - 1}{1 - \gamma} \tag{2.4}$$

## 2.2. IPO Scenario

The following section describes the scenario the entrepreneur faces when making the decision about when to IPO his company. Suppose we are at a time prior the IPO. Let $(C_{it})_t$ be the process of consumption of the agents $i = (1, 2)$ (entrepeneur and market resp.). Before the IPO the economy is segmented, and the agents can only consume the dividend payoffs of their respective assets. The entrepreneur has the option to decide when he will make his company public. Let $\tau$ be the optimal time chosen by $E$. We must first determine conditions over entrepreneur's wealth at $\tau$.

### 2.2.1. IPO Conditions

Suppose that at the IPO time $E$ faces transactional costs $CT$. These should include any costs perceived by the entrepreneur, e.g. administrative costs, losing private control benefits (Benninga et al., 2005), legal costs, etc. We will suppose $CT$ is proportional to the economy's overall wealth, i.e. $CT = \alpha(P_1 + P_2)$. This simple assumption is partially based in empirical facts documented in Ritter (1987), and on the costs of losing private benefit control from Benninga et al. (2005). Fixed costs can be assumed as a fraction of overall wealth, and variable and control costs can be assumed proportional to the IPO gross proceeds. Transactional costs should be function of $P_1$ and $P_2$ (homogeneous of degree 1). We can easily extend the model to consider a more general transaction cost structure (i.e. $CT = \alpha_1 P_1 + \alpha_2 P_2$ for $\alpha_1$,$\alpha_2 \in \mathbb{R}$), but we prefer to keep the model simple.

His interchangeable wealth will be given by the price the market is willing to pay for $P_1$ minus $CT$. Let $(W_{it})_t$ be the agent's process $i = (1, 2)$ (entrepeneur and market resp.). Writing down this conditions:

$$W_{1\tau} = P_{1\tau} - \alpha(P_{1\tau} + P_{2\tau}) \tag{2.5}$$

The market agent interchangeable wealth is still given by $W_{2t} = P_{2t}$, which means that the aggregate wealth in $\tau$ is reduced to $W_{1\tau} + W_{2\tau} = (1-\alpha)(P_{1\tau} + P_{2\tau})$. The aggregate wealth is now given by two assets minus a fraction. For being able to trade, a permanent cost is incurred, and the overall economy suffers a wealth reduction. For any $t > \tau$, the following relation must hold:

$$W_{1t} + W_{2t} = (1 - \alpha)(P_{1t} + P_{2t}) \tag{2.6}$$

At $\tau$, the asset is incorporated to the market portfolio. Once the IPO is made, the market changes, and the economy prices the assets as in Cochrane et al. (2007) and Martin (2007). Next chapter describes in detail asset pricing (including agent's wealth) in this scenario: $P_{it}/D_{it}$ is a function of $s_t$, $W_{1t} = \omega_\tau (P_{1t} + P_{2t})(1-\alpha)$, and $W_{2t} = (1-\omega_\tau)(P_{1t} + P_{2t})(1-\alpha)$, where $\omega_\tau = \frac{C_{it}}{C_{1t}+C_{2t}}$ for any $t \geq \tau$. This means that at the IPO time both agents rebalance their positions and end up holding the new market portfolio. Since we know the wealth of each agent in $\tau$, it is easy to verify that

$$\omega_\tau = \frac{P_{1\tau} - \alpha(P_{1\tau} + P_{2\tau})}{(P_{1\tau} + P_{2\tau})(1-\alpha)} \tag{2.7}$$

As shown in next chapter, (3.4) and (3.13) leads us to the following relation for the consumption of $E$ and $M$ for $t \geq \tau$:

$$C_{1t} = \omega_\tau (1-\alpha)(D_{1t} + D_{2t}) \tag{2.8}$$

$$C_{2t} = (1-\omega_\tau)(1-\alpha)(D_{1t} + D_{2t}) \tag{2.9}$$

At any time $t > \tau$, the agents consume the same constant fraction of the aggregate consumption as in $\tau$.

### 2.2.2. Optimal IPO Timing

As we have shown, consumption stream is a function depending on the processes $(D_{is})_s$ $(i = 1, 2)$. If the IPO is exercised at $\tau$, the entrepreneur's indirect utility at $t < \tau$ will be given by

$$J^\tau(D_{1t}, D_{2t}) = \mathbb{E}_t\left[\int_t^\infty e^{-\delta(s-t)} u(C_{1s})\, ds\right] \tag{2.10}$$

$$= \mathbb{E}_t\left[\int_t^\tau e^{-\delta(s-t)} u(D_{1s})\, ds + e^{-\delta(\tau-t)} \int_\tau^\infty e^{-\delta(s-\tau)} u(C_{1s})\, ds\right]$$

$$= \mathbb{E}_t\left[\int_t^\infty e^{-\delta(s-t)} u(D_{1s})\, ds + j_E(\tau - t, D_{1\tau}, D_{2\tau})\right] \tag{2.11}$$

Where $j_E(\tau - t, D_{1\tau}, D_{2\tau}) = e^{-\delta(\tau-t)}\mathbb{E}_\tau\left[\int_\tau^\infty e^{-\delta(s-\tau)}\left(u(C_{1s}) - u(D_{1s})\right) ds\right]$. The first term in the right side of equation (2.11) is the expected utility of the entrepreneur if he never makes the IPO. The second term is the extra value the IPO option adds to the utility. Since the first term does not depend on $\tau$, maximizing $J^\tau(D_{1t}, D_{2t})$ is equivalent to maximize $\mathbb{E}_t\left[j_E(\tau, D_{1\tau}, D_{2\tau})\right]$. The appendix shows that $j_E$ can be obtained in closed form, and is given by an expression of the form $j_E(\tau - t, x, y) = e^{-\delta(\tau - t)} x (xy)^{-\gamma/2} \left( j(s, \omega_\tau) - \frac{\left(\frac{1-s}{s}\right)^{\gamma/2}}{\delta - c(1-\gamma, 0)} \right)$, with $s = \frac{x}{x+y}$ and $\omega_\tau$ as shown above. The optimal IPO decision the entrepreneur faces is represented by the following *optimal stopping problem*:

Find the *value function* $V(t, x, y)$ and $\tau^*$ such that

$$V(t, x, y) = \sup_{\tau} j_E(\tau - t, x, y) = j_E(\tau^* - t, x, y) \tag{2.12}$$

For solving this problem, we base our procedure on Øksendal & Sulem (2005) using the HJB method. The *value function* will be given by the following proposition. Further details can be found in the Appendix.

PROPOSITION 2.1. *The value function of the entrepreneur is given by a function $\phi(t, x, y)$ and an stopping time $\tau_D$ satisfying the following conditions. Suppose a function $H$ : $[0, 1] \to \mathbb{R}$ and constants $0 \le s_1^* < s_2^* \le 1$ such that:*

*$H(s) \in C^1((0, 1)) \cup C([0, 1]) \cup C^2((0, 1) \setminus \partial D)$ with $D = \{s \in (0, 1) \backslash H(s) > j(s, \omega_\tau)\}$ and*

$$H(s) = \begin{cases} h_1(s) & s < s_1^* \\ j(s, \omega_\tau) - \frac{\left(\frac{1-s}{s}\right)^{\gamma/2}}{\delta - c(1-\gamma, 0)} & s_1^* < s < s_2^* \\ h_2(s) & s > s_2^* \end{cases} \tag{2.13}$$

*With*

$$h_i(s) = c_i \left( \frac{1-s}{s} \right)^{\lambda_i} \qquad i = 1, 2 \tag{2.14}$$

*For some constants $\lambda_1 < 0 < \lambda_2$ such that they solve equation (A.13), and $c_i$ such that $D$ is given by $D = [0, 1] \setminus [s_1^*, s_2^*]$. Let $\tau_D := inf\{t > 0 \backslash s \notin D\}$. Then $\phi(t, x, y) = e^{-\delta t} x (xy)^{-\gamma/2} H(s)$ and $\tau^* = \tau_D$ (under some conditions over the processes).*

PROOF. The proof is based in solving the homogeneus PDE associated with the generator of the process $Z_s = (t + s, D_{1s}, D_{2s})^T$, with $Z_0 = (t, x, y)$, verifying then optimality conditions. For further details see the Appendix. □

This proposition shows that the optimal IPO strategy will be determined by two IPO triggers, the *Small cap IPO Trigger* $s_1^*$ and the *Large cap IPO Trigger* $s_2^*$. It may exist the case in which $s_1^* = 0$ or/and $s_2^* = 1$. The important thing is that $E$ will IPO when the state process $s_t$ is between this triggers. The so called *value matching* and *smooth pasting* conditions are implicitly given by last proposition ($H(s)$ must be $C^1((0,1)) \cup C([0,1])$). The following corollary gives us a direct method for obtaining the constants $c_i$ and the optimal states $s_i^*$ for ($i = 1, 2$) based on last proposition.

**Corollary 2.1.** *The optimal states $s_i^*$ can be obtained by solving the following equations:*

$$s_i^*(1-s_i^*)\left(j(s_i^*,\omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{\gamma/2}}{\delta - c\,(1-\gamma,0)}\right)' + \lambda_i\left(j(s_i^*,\omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{\gamma/2}}{\delta - c\,(1-\gamma,0)}\right) = 0$$

$$i = 1,2 \tag{2.15}$$

*and $c_i$ is given explicitly by the expression*

$$c_i = \frac{j(s_i^*,\omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{\gamma/2}}{\delta-c(1-\gamma,0)}}{\left(\frac{1-s_i^*}{s_i^*}\right)^{\lambda_i}} \quad i = 1,2 \tag{2.16}$$

# 3. ASSET PRICING

The first half of this chapter includes some results that must be used to solve the model presented in previous chapter, i.e. prices and wealths in equilibrium. The second half is used in chapter 4, specifically in before v/s after IPO and optimal Beta analysis. Most of the following arise from the work of Martin (2007), who generalizes the asset pricing in Lucas Trees economies developed by Cochrane et al. (2007). Some equilibrium conditions must be stated so that unique prices exists with heterogeneous agents, resembling the procedure used by Longstaff (2009).

## 3.1. Asset's Prices in Complete Markets

First, suppose the IPO decision is irreversible, and done at some time $\tau$. At the IPO time, the market completes and both agents rebalance their portfolio. Each of the agents can choose freely in $t$ how much to consume and how much to invest in $P_1$ and $P_2$. After this happens, the market is in equilibrium, and the scenario resembles the *two lucas trees* economy Cochrane et al. (2007).

### 3.1.1. Asset Prices in Equilibrium

The first order conditions for the asset prices, derived by Lucas (1978), states that from the point of view of the entrepeneur

$$P_{it} = \mathbb{E}_t \left[ \int_t^{\infty} e^{-\delta(s-t)} \left( \frac{C_{1s}}{C_{1t}} \right)^{-\gamma} D_{is} \, ds \right] \tag{3.1a}$$

and from the point of view of the market

$$P_{it} = \mathbb{E}_t \left[ \int_t^{\infty} e^{-\delta(s-t)} \left( \frac{C_{2s}}{C_{2t}} \right)^{-\gamma} D_{is} \, ds \right] \tag{3.1b}$$

Since the market is complete, state-price deflators must be unique and prices form both points of view must be the same. It follows that

$$e^{-\delta(s-t)} \left( \frac{C_{1s}}{C_{1t}} \right)^{-\gamma} = e^{-\delta(s-t)} \left( \frac{C_{2s}}{C_{2t}} \right)^{-\gamma} = \xi_{s,t} \tag{3.2}$$

We can isolate $\xi$ in terms of both consumption streams, obtaining:

$$\xi_{s,t} = e^{-\delta(s-t)} \left( \frac{C_{1s} + C_{2s}}{C_{1t} + C_{2t}} \right)^{-\gamma} \tag{3.3}$$

We should also notice that the law of unique prices will also imply that for any $t, s \geq \tau$

$$\frac{C_{is}}{C_{1s} + C_{2s}} = \frac{C_{it}}{C_{1t} + C_{2t}} = \omega_\tau \tag{3.4}$$

Which means that the fraction consumed from the aggregate consumption stream remains constant, and we are calling this constant $\omega_\tau$.

In the cases where

$$\frac{C_{1s} + C_{2s}}{C_{1t} + C_{2t}} = \frac{D_{1s} + D_{2s}}{D_{1t} + D_{2t}} \tag{3.5}$$

asset prices are given by the following proposition, in line with the results of Martin (2007). We will prove later that (3.5) holds because of the transactional costs assumptions.

PROPOSITION 3.1. *Under the above conditions, the asset prices in equilibrium are given by*

$$P_{it} = \frac{D_{it}}{\sqrt{s_t^\gamma (1-s_t)^\gamma}} \int_{-\infty}^{\infty} \left( \frac{1-s_t}{s_t} \right)^{iv} \frac{\Psi_\gamma(v)}{\delta - c_i(v)} \, dv \tag{3.6}$$

*Where*

$$\Psi_\gamma(v) = \frac{1}{2\pi} \frac{\Gamma(\gamma/2 - iv)\Gamma(\gamma/2 + iv)}{\Gamma(\gamma)} \tag{3.7}$$

$$c_1(v) = \mathbf{c}(1 - \gamma/2 - iv, -\gamma/2 + iv) \tag{3.8}$$

$$c_2(v) = \mathbf{c}(-\gamma/2 - iv, 1 - \gamma/2 + iv) \tag{3.9}$$

*and* $\mathbf{c}(\theta)$ *the CGF of the process* $y_{t+1} - y_t$

PROOF. The prices can be obtained calculating the conditional expectation of (3.1a) and (3.1b) using the state-price deflator (3.3) using the assumption (3.5). Procedure is the same as in Martin (2007). Details can be found in the Appendix. □

### 3.1.2. Agents' Wealth and Consumption in Equilibrium

Let $W_{it}$ $(i = 1, 2)$ be the wealth of the agents in equilibrium (entrepreneur and market resp.). In an exchange economy today's wealth must equal the total discounted consumption streams in the future, i.e.

$$W_{it} = \mathbb{E}_t \left[ \int_t^\infty \xi_{t,s} C_{is} \, ds \right] \tag{3.10}$$

Replacing 3.4 in 3.10:

$$W_{it} = \frac{C_{it}}{C_{1t} + C_{2t}} \mathbb{E}_t \left[ \int_t^\infty \xi_{t,s} \left( C_{1s} + C_{2s} \right) \, ds \right] \tag{3.11}$$

Using this equation, (2.6), and the fact that $P_{it} = \mathbb{E}_t \left[ \int_t^\infty \xi_{t,s} D_{is} \, ds \right]$

$$\mathbb{E}_t \left[ \int_t^\infty \xi_{t,s} \left( C_{1s} + C_{2s} \right) \, ds \right] = \mathbb{E}_t \left[ \int_t^\infty \xi_{t,s} (1-\alpha)(D_{1s} + D_{2s}) \, ds \right] \tag{3.12}$$

Since this equation is satisfied for any $t > \tau$, for any $t_1, t_2 > \tau$ we have that $\mathbb{E}_t \left[ \int_{t_1}^{t_2} \xi_{t,s} \left( C_{1s} + C_{2s} \right) - (1-\alpha)(D_{1s} + D_{2s}) \, ds \right] = 0$, which means that for any $t > \tau$ the aggregate consumption is given by

$$C_{1t} + C_{2t} = (1-\alpha)(D_{1t} + D_{2t}) \tag{3.13}$$

This is the new market clearing condition in equilibrium. This proves (3.5) statement and we can price the economy assets using Martin (2007).

Using (3.11) and (3.13), and noticing (3.4) we see that under equilibrium conditions, the economy agents' wealth process in terms of asset prices can be determined with the consumption share $\omega_\tau = \frac{C_{1t}}{C_{1t}+C_{2t}}$ at any time t, i.e.

$$W_{1t} = \omega_\tau \left( P_{1t} + P_{2t} \right) (1-\alpha) \tag{3.14}$$

$$W_{2t} = (1-\omega_\tau) \left( P_{1t} + P_{2t} \right) (1-\alpha) \tag{3.15}$$

## 3.2. Asset's Returns in Complete Markets

We will proceed with the study of asset's returns and risk free returns. This will permit us to analyze risk factors faced by the entrepreneur.

### 3.2.1. Risk-Free Return

For obtaining the instant risk-free return in equilibrium, we will base our procedure on Martin (2007). Suppose we are at a time $t$. First, we price a bond $B_T$ that pays a unit in $T > t$. Then we calculate the dividend yield $R_{t\to T}$ using the fact that $B_T = e^{-R_{t\to T}(T-t)}$. Finally we can obtain the instant risk-free return with the limit of the dividend yield as $T$ tends to $t$, i.e. $r_t^f = \lim_{T\to t} R_{t\to T}$. Details of this procedure can be found on the Appendix. The result is shown below.

$$r_t^f = \frac{1}{\sqrt{s^\gamma(1+s)^\gamma}} \int_{-\infty}^{\infty} \left(\frac{1-s}{s}\right)^{iv} (\delta - \mathbf{c}(-\gamma/2 - iv, -\gamma/2 + iv))\, \Psi_\gamma(v)\, dv \tag{3.16}$$

### 3.2.2. Risky Asset's Returns

Let $(r_t^i)_t$ be the asset's return for $i = (1,2)$. By definition, the asset's total return is given be the *capital gains* plus the *dividend yield*, i.e.

$$r_{it}\, dt = \frac{dP_{it}}{P_{it}} + \frac{D_{it}}{P_{it}} dt \tag{3.17}$$

Following Martin (2007), we can find a close expression for the expected returns. Using expressions (3.17), the *dividend yield* is given by the reciprocal of the $Price/Dividend$ obtained before. The only things left to calculate are the *capital gains*, specifically $\mathbb{E}_t\left[dP_t\right]$. The procedure is exactly the same as in Martin (2007). Details can be found in the Appendix. Finally

$$\mathbb{E}_t\left[r_{it}\right] = \frac{\sum_{n=0}^{\infty} \binom{\gamma}{n} \left(\frac{1-s_t}{s_t}\right)^{-n} \int_{-\infty}^{\infty} \left(\frac{1-s_t}{s_t}\right)^{iv} h_i(v)\mathbf{c}(w_{in})\, dv}{\sum_{n=0}^{\infty} \binom{\gamma}{n} \left(\frac{1-s_t}{s_t}\right)^{-n} \int_{-\infty}^{\infty} \left(\frac{1-s_t}{s_t}\right)^{iv} h_i(v)\, dv} + \frac{D_{it}}{P_{it}} \tag{3.18}$$

Where

$$h_i(v) = \frac{\Psi_\gamma(v)}{\rho - c_i(v)} \tag{3.19}$$

$$w_{1n} = (1 - \gamma/2 + n - iv, \gamma/2 - n + iv) \tag{3.20}$$

$$w_{2n} = (\gamma/2 + n - iv, 1 + \gamma/2 - n + iv) \tag{3.21}$$

### 3.3. Asset Pricing Before the IPO

Before the IPO the entrepreneur faces an incomplete market because his asset is illiquid. We cannot assume there exists unique state price deflators. Entrepreneur might value his company in a different way the market does, mainly because there is an option involved in the IPO timing. We shall price using an indirect utility function approach.

Let us define the company's price from the entrepreneur's point of view as the minimum price at which the $E$ is willing to IPO his company. This means that at this price the entrepreneur's utility before and after the IPO must be the same. Under this assumptions, we can price using an indirect utility function approach, choosing a price at which the value function is $C([0,1])$. We can resume this by the following statement. Suppose $P^*_{1s}$ is the price at which the entrepreneur is indifferent to IPO the company. For $s \in \{s^*_1, s^*_2\}$ we have that $P^*_{1s} = P_{1s}$. Outside this region, we can isolate the price from the following relations:

$$\begin{cases} h_1(s) = j(s, \omega{*}_\tau) & s < s^*_1 \\ h_2(s) = j(s, \omega{*}_\tau) & s > s^*_2 \end{cases} \tag{3.22}$$

Where $\omega{*}_\tau = \frac{P^*_1 - \alpha(P_{1s} + P_{2s})}{(P_{1s} + P_{2s})(1-\alpha)}$. $h_i$ are as in Proposition 2.1. The entrepreneur agent is not able to change transactional costs and aggregate wealth even if he values more his asset, so we keep them the same as in $\omega_\tau$.

## 4. RESULTS AND ANALYSIS

In the following section we study the main implications of our model. Although the theoretical model accounts for jumps in the dividend processes, we consider only the diffusion component for most of our analysis. A simpler model is desirable to highlight the main contributions of the paper and the economic intuition behind the results. First we analyse the behaviour and sensibility of optimal IPO states. We study how dividends correlation and transactional costs affect in the IPO decision. Expected IPO times sensibility to initial dividend sizes is also studied. Then, we analyze the diversification effect on both, entrepreneur and market. In order to study the aggregate economy's optimal IPO trigger, we solve the optimal timing considering both agents. We go a little further and solve the model introducing the effect of taxes and subsidies, in order to maximise aggregate utility. In addition we study the relations between optimal IPO triggers and Betas of the newly listed firm. Finally we consider the effect of jumps in dividends and analyze their effect on the IPO triggers.

### 4.1. Optimal IPO Triggers

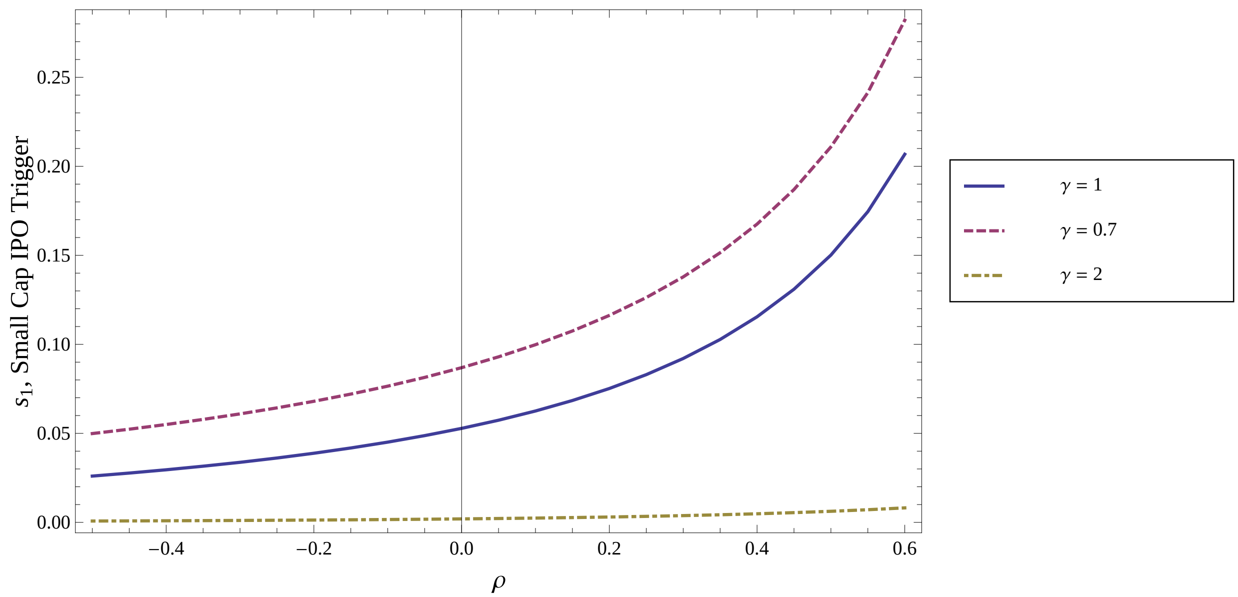
In this section we study in detail the market conditions at which the entrepreneur IPOs his asset. As mentioned above, the optimal IPO strategy is determined by a closed interval between dividend ratios $s_1^*$ and $s_2^*$: the *small-cap IPO trigger* and the *large-cap IPO trigger*.

When the private company has a small share of the economy, i.e. $s < s_1^*$, it will be optimal to postpone the decision. Although the IPO diversifies the entrepreneur's consumption, transactional costs are too big. Utility gains due to diversification does not compensate the cost to IPO. The optimal IPO strategy is to wait until the company owns a bigger share of the economy, which implies greater diversification benefits (diversification finds its maximum for middle values of s, as seen in Cochrane et al. (2007)).

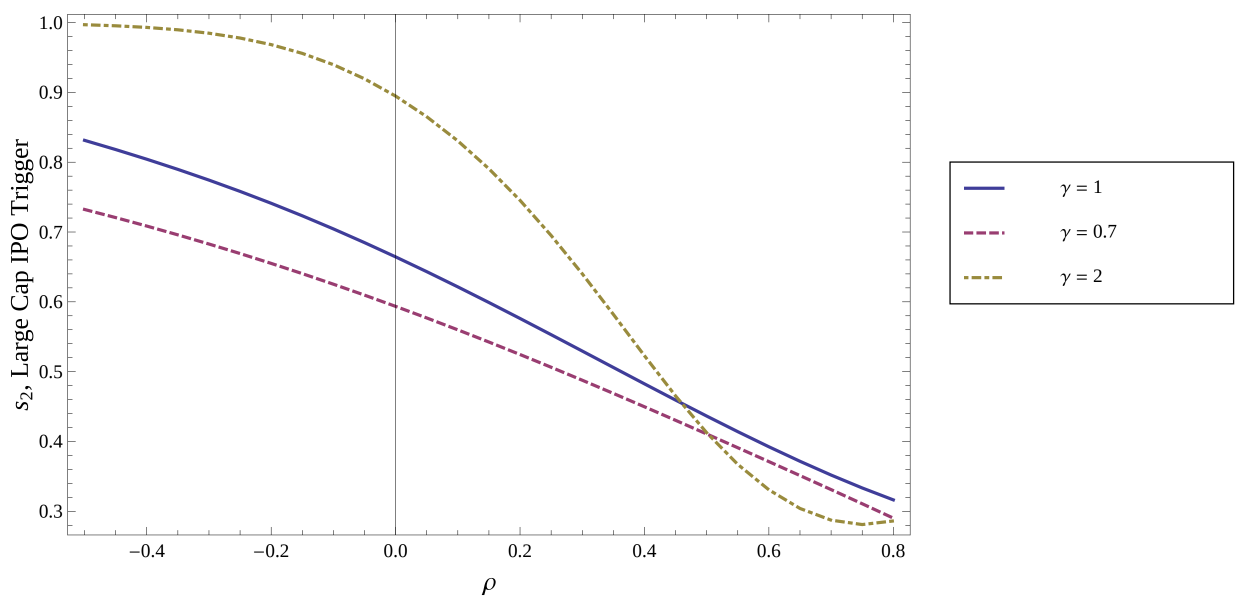
In the unusual case in which the company owns a large share of the economy, i.e. $s > s_2^*$, the entrepreneur is not tempted to IPO its company: The potential market portfolio after the IPO will be too correlated with the company. Diversification benefits are small comparing to the variable transactional costs (for values of s near $0$ or $1$ diversification is minimal). The entrepreneur will keep its company private until its share in the economy diminishes (produced by negative shocks to the company's dividends, or positive shocks to the market's dividends). This situation sounds absurd in mature economies, but may occur in an early stage of less developed economies, when only a few firms are listed in the stock market.

Figure 4.1 presents the IPO Triggers for different correlations between dividend streams $D_1$ and $D_2$, for different values of risk aversion of the agents. Figure 4.1(a) shows that the *small-cap IPO trigger* increases with the correlation of the dividend streams and decreases with risk aversion. Diversification benefits of the IPO diminishes if the company is correlated with the market, which means the entrepreneur will wait a larger market share of his asset, increasing the optimal IPO trigger. High risk averse entrepreneur agents show low sensitivity towards dividend's correlation, and a low market share is sufficient to IPO. In this cases diversification is highly desirable. In this case before the IPO means that $s < s_1^*$: higher $s_1^*$ should delay entrepreneur's decision.

A partially symmetric effect is reflected in Figure 4.1(b). The *large-cap IPO trigger* mostly decreases with the correlation and its sensitivity increases with risk aversion. If the entrepreneur agent holds a large fraction of the economy (i.e. $s > s_2^*$), a higher correlation with the market decreases its diversification benefits. Lower negative correlations increases diversification benefits, and high risk averse agents are willing to IPO at higher market shares. Variable costs, such as the lose of private control shown in Zingales (1995) and Benninga, Helmantel, & Sarig (2005), gains importance in the IPO decision in large companies. Higher risk aversion and lower dividend's correlation incentivates to IPO at a higher share because diversification is more desirable. Therefore, the entrepreneur will delay the IPO decision until $s < s_2^*$, i.e. diversification benefits are enough to compensate the variable transactional costs. A private large company means that $s > s_2^*$, hence lower $s_2^*$ should delay the IPO decision. Although the existence of this *large-cap IPO trigger* is a valid result of the model, from now on we will exclusively concentrate on the Small Cap IPO Trigger, being the most common case faced in economies.




(a) Small Cap IPO Trigger versus $\rho$




(b) Large Cap IPO Trigger versus $\rho$

FIGURE 4.1. Optimal IPO triggers with respect to the correlation between both dividend payoffs, $\rho$. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25\sqrt{1-\rho 2}$, $\sigma_{12} = 0.25\rho$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, and $\alpha = 0.05$.

Let's now focus our attention on the behaviour of the IPO triggers against movements of the transactional costs. Figure 4.2 shows that the behaviour of IPO triggers against transactional costs is similar as against dividend's correlation (shown in figure 4.1(a)). Results

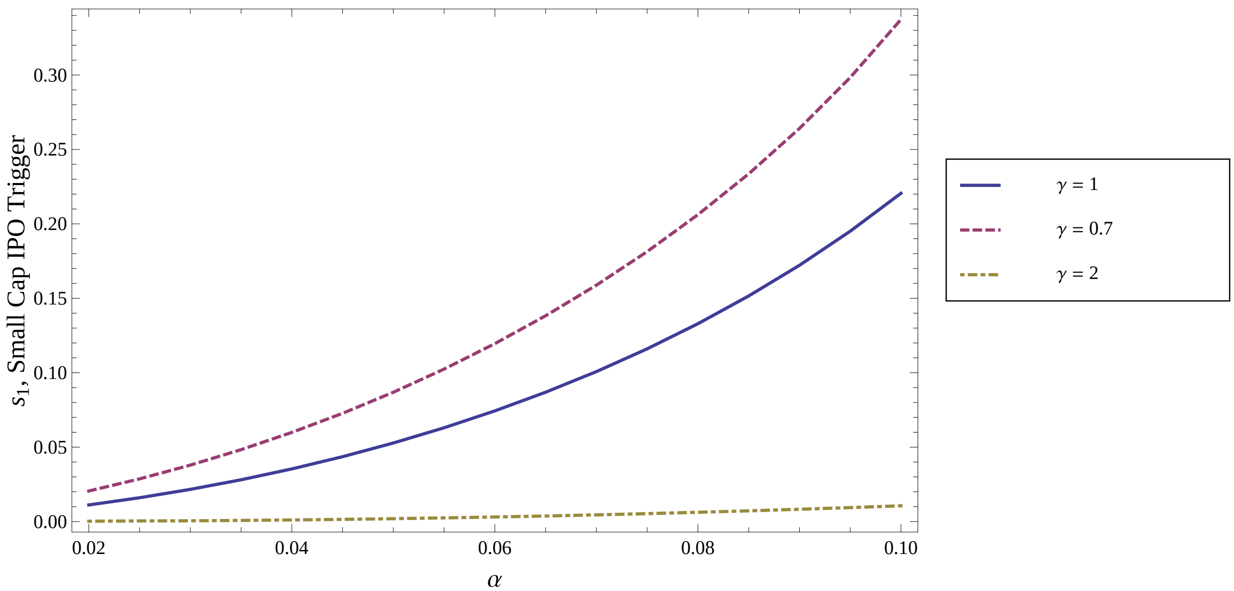



FIGURE 4.2. Small-cap trigger with respect to the fixed transaction costs $\alpha$. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$ and $\rho = 0$.

supports intuition: the higher the transactional costs, the larger the size of the company required to IPO. In this cases, an increase in $\alpha$ produces an important rise in fixed costs (proportional to market). For small companies, these costs play a main role in determining the optimal IPO trigger, rapidly increasing their market share required to IPO. For lower risk aversion, the model is highly sensitive to transactional costs, and shows a convex behaviour. High risk averse agents are less sensitive to transactional costs, and IPO at lower shares even if the transaction costs are high: Diversification strongly increases their utility function, and smaller market shares compensate transactional costs.

### 4.2. Expected Hitting Time

In this section we study the expected time remaining until the IPO option is exercised. Studying the expected time remaining to IPO might be more intuitive than studying the optimal state processes at which the IPO is triggered.

Expected hitting time is calculated using a monotonic transformation of $s_t$: $u_t = \log\left(\frac{1-s_t}{s_t}\right) = y_{2t} - y_{1t}$. Random variable $u_t : \Omega \to \mathbb{R}$ distributes the same as variables $y_{it}$, i.e. it follows jump diffusion processes.

It can be proved that expected time to IPO when processes $y_{it}$ have the same drift is infinite (see Shreve (2004)). An intuitive explanation might be that because the expected growth company equals market's growth, the firm is never expected to reach the optimal relative size to IPO. To calibrate the variables, we chose different drifts for dividends. We assume that the firm's dividends grow faster than the market dividend payoffs.

For obtaining the distribution of the stopping time variable $\tau_D = \inf\{t > 0 \backslash \text{The company stays private}\}$ we use the results obtained by Coutin & Dorobantu (2009). Knowing the distribution, expectations can be easily computed.

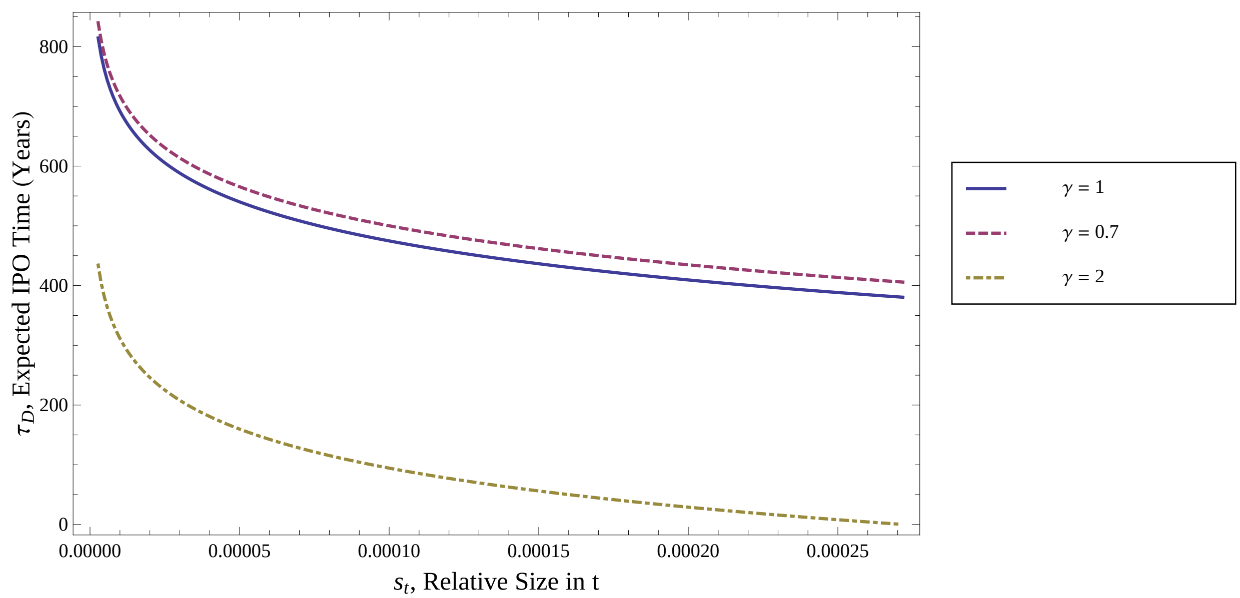



FIGURE 4.3. Expected IPO time as a function of the initial relative dividend. The parameters used are $\mu_{D_1} = 0.07$, $\mu_{D_2} = 0.04$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\rho = 0$ and $\alpha = 0.05$.

Figure 4.3 shows the behaviour of expected hitting time to IPO against different initial sizes (relative to the market). For companies with lower $s$, expected hitting times are huge. This expectation lowers when the firm's size approaches the optimal $s$ to IPO. This means that the smaller the IPO trigger, the lower is the expected time a company has to wait to

IPO. Entrepreneurs with low risk aversion are expected to wait more time to IPO their assets.

Assuming that firms within an industry have similar dividend streams, the model predicts that IPO occurrences will be positively correlated with firms' size. This means that the first IPOs will be triggered by larger companies in an industry, and then smaller firms will follow. This facts are empirically documented in Chemmanur et al. (2009).

### 4.3. Welfare Analysis

This section studies the effect of diversification contrasting scenarios before and after the IPO. First we analyse the perceived effect of the timing option over the utility of the agent's involved in the economy (entrepreneur and market). We proceed studying the effect over the firm's value considering indirect utility valuation used in (3.22). Finally we analyse how the economy agent's consumption changes with the IPO and the new diversified scenario.

#### 4.3.1. Economy Agent's Utilities

The entrepreneur initial optimal control problem is determined by the stopping time $\tau$ that maximizes his utility function. This option gives him the liberty of choosing when to IPO, moment in which the whole economy wealth scenario will be determined. Recall that the initial distribution of wealth after the IPO is given by $\omega_\tau$ in (2.7). This fraction remains constant after the IPO, as stated in (3.4). We wish to analyse the behaviour of both agents' utility function, and study the welfare effect of this IPO option.

To calculate each agent's utility function before the IPO, we use the Feynman-Kac approach to solve the following conditional expectation:

$$V^i\left(D_{1t}, D_{2t})\right) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\delta(s-t)} u(C_{is})\, ds\right] \tag{4.1}$$

$$= \mathbb{E}_t\left[\int_t^{\tau} e^{-\delta(s-t)} u(D_{is})\, ds + \int_t^{\tau} e^{-\delta(s-t)} u(C_{is})\, ds\right] \tag{4.2}$$

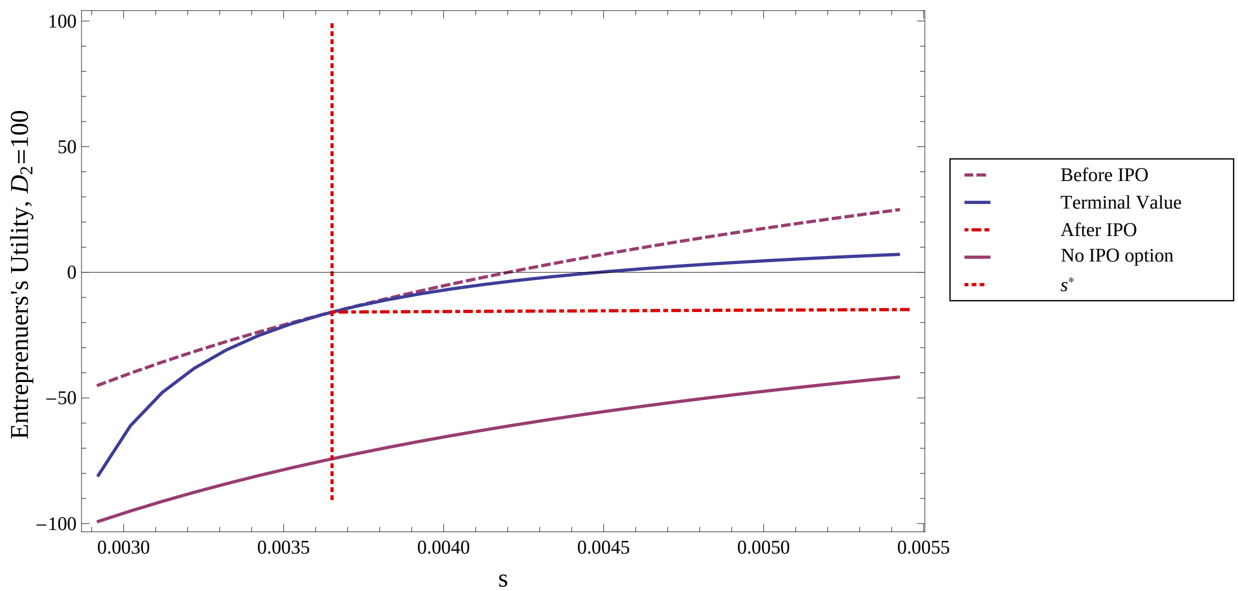
After the IPO, the utility for agent $i$ will be given by

$$V^i\left(D_{1t}, D_{2t}\right) = \mathbb{E}_t\left[\int_t^\infty e^{-\delta(s-t)} u(C_{is})\, ds\right] = \frac{1}{1-\gamma}\left(\frac{W_{it}}{C_{it}^\gamma} - \frac{1}{\delta}\right) \tag{4.3}$$

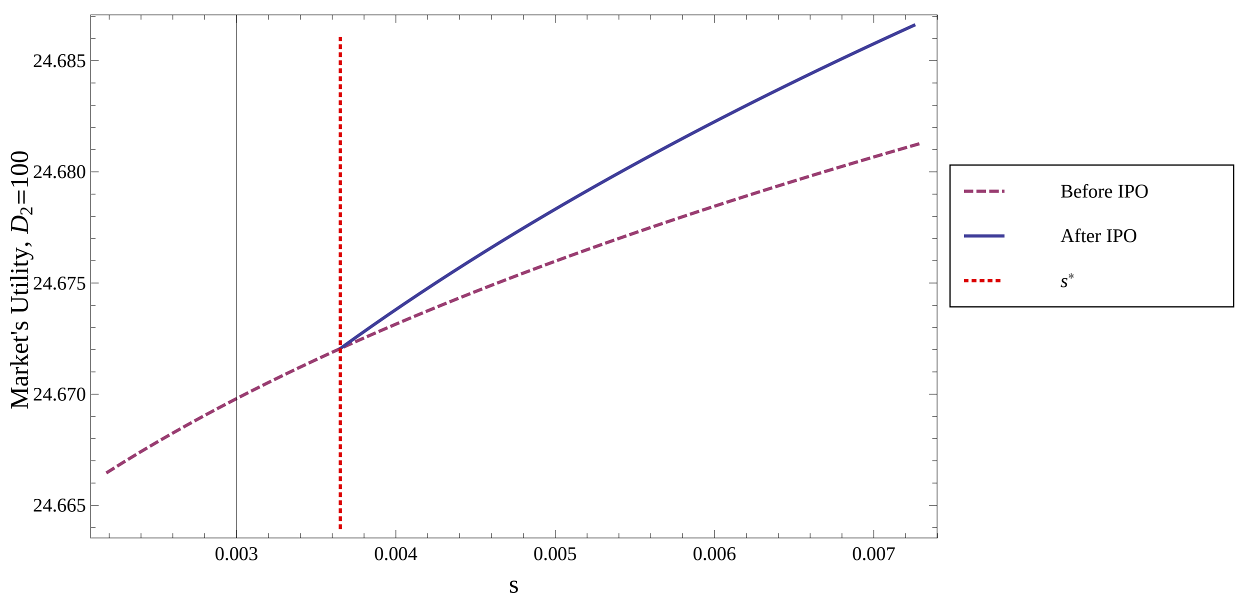
where $C_{it}$ is the consumption in equilibrium after the IPO, given by (2.8). $W_{it}$ is wealth in equilibrium, and is given by (3.14). For details on the resolution of (4.1) and (4.3) please see Appendix F.

Figure (4.4) shows the behaviour of each agent's utility function, fixing $D_2$ in $100$ to reduce degrees of freedom in the analysis. Figure 4.4(a) shows several information. The line labelled ”Before IPO” refers to the entrepreneur's utility before the IPO. We notice that this line is tangent and touches the thick blue line at the IPO trigger, which is the utility of the entrepreneur at the IPO time (plotted against different IPO triggers). This ”value matching” and ”smooth pasting” conditions are required so that the value function represents an optimal solution to the agent's problem. The thick blue line differs from the utility after the IPO because the fraction of the aggregate consumption gets fixed at the IPO time, and will only match at the IPO. The line labelled ”No IPO option” represents that entrepreneur utility if we remove the IPO option. We clear see the added value the IPO timing option gives to the entrepreneur, even though in this case there is no transactional costs involved. The merely presence of the IPO timing option makes the entrepreneur ”better”.

Figure 4.4(b) shows the behaviour of market's utility for different values $s$. Although the market does not control the IPO timing, he is aware of when the entrepreneur will optimally make his company public. We see that the utility before the IPO increases in $s$, although we may think that the growth of an unlisted company should not affect directly on market's utility. This increase of utility is given by higher chances that the company IPOs, diversifying consumption and making the whole market ”better”. We see there are no jumps in the utility at the IPO time, but there is an increase of the slope as the whole market after the IPO becomes more diversified. The market gains diversification without paying transactional costs, and is able to access a greater fraction of the aggregate consumption, which will rapidly increase his utility as the newly listed company market share grows, diversifying the new market.




(a) Entrepreneur's Utility




(b) Market's Utility

FIGURE 4.4. Agents' Utility. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$,$\rho = 0$, $\alpha = 0.05$, and $D_2 = 100$.

### 4.3.2. Price/Dividend Ratio

We now compare the market price-dividend versus the one in which the entrepreneur is willing to IPO. If the entrepreneur IPOs at a lower share, IPO incomes minus transactional does not generate enough diversification benefits. The agent will not IPO at a lower share because the asset's market value is too low. Our purpose is to study the price at which the entrepreneur would IPO when market shares are under the optimal trigger. The difference between this price and the actual market price is a good indicator to measure how much it costs to turn an illiquid asset into a liquid one.

Before the IPO the asset is not tradable in the market. Therefore, we cannot assume complete markets in its valuation (The uniqueness of the pricing kernel is not assured). The entrepreneur might value his asset in a different way the market does, mainly because of the IPO timing option: The agent may wait for better conditions to trigger the IPO. We use a utility indifference approach to value the unlisted firm from the entrepreneur's point of view, as described in chapter 3. We defined in equations (3.22) the company's price at which the entrepreneur is willing to IPO. Scaling this result by its dividend process $D_{1t}$, we obtain an expression that is function of our state process $s_t$. Once the IPO is done, the asset becomes tradable in a complete market and there is a unique pricing kernel.

Figure (4.5) shows the IPO timing option in terms of price-dividend ratios. Before the IPO, the entrepreneur will only IPO at a higher price than the actual market value of his asset. We may consider this difference as the costs to liquidate the firm. From the entrepreneur's point of view, the tradable asset has a higher price than the illiquid asset before the optimal IPO timing. In this case, market's valuation is based in the discounted cash flows of the liquid asset. Because the market is willing to pay a lower price that the one required by the company owner, the IPO will not be triggered, and the asset will remain illiquid. When the firm's relative size approaches the IPO trigger, the extra value required to make the asset liquid decreases until it reaches zero. The IPO incomes perceived by the entrepreneur generate enough diversification benefits, and the IPO will be triggered.

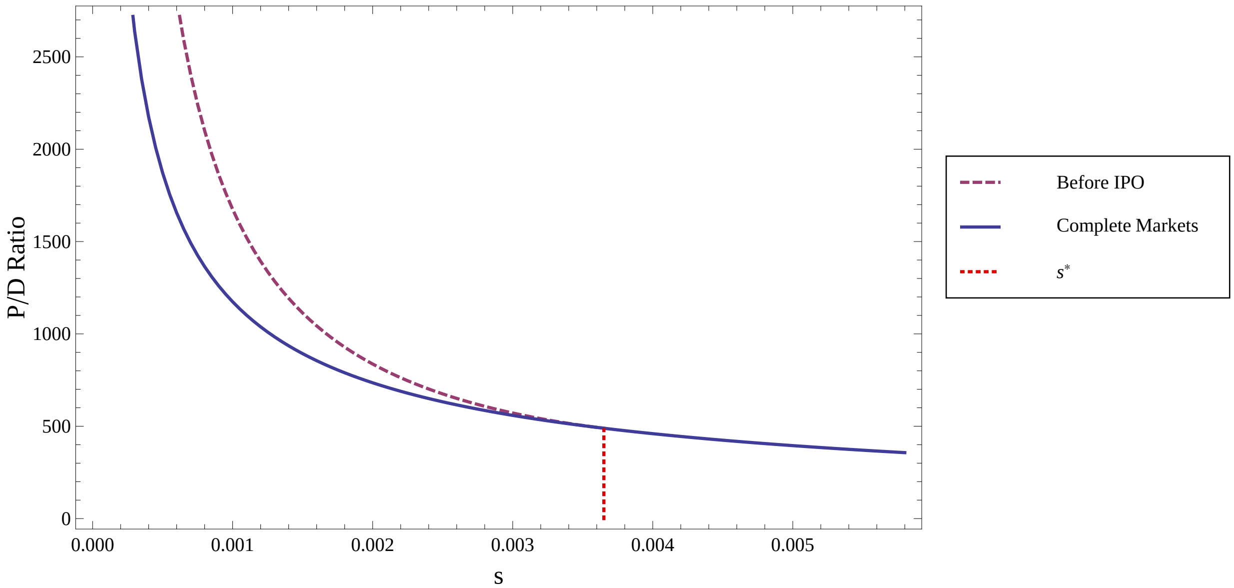



FIGURE 4.5. Price/dividend ratios for entrepreneur's asset. The dashed line shows the price/dividend ratio required to IPO, the solid line the price/dividend from market's point of view, and the red dashed line shows the optimal IPO trigger. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$, $\rho = 0$ and $\gamma = 2$.

### 4.3.3. Entrepenuer's Consumption

For a better understanding of the entrepreneur's decision motives, we need to study the consumption streams, which becomes the core process of the utility function. Before the IPO the entrepreneur and market can only consume the dividends generated by its assets. After the IPO, as stated in equations (2.8), both entrepreneur and market owns a diversified portfolio containing both assets. This new asset portfolio produces a dividend stream that is proportional to $D_1 + D_2$, which reduces the volatility stream of the goods used for consumption. The whole economy is benefit by this change in consumption streams if optimally triggered by the entrepreneur.

To analyse this effect, let us simplify the dividend processes and assume that are no discontinuous jumps in the stochastic processes. In this scenario dividend streams follows geometric brownian motions. As shown in (2.8), in equilibrium we have $C_{1t} = \omega_\tau (D_{1t} + D_{2t})$ and $C_{2t} = (1 - \omega_\tau)(D_{1t} + D_{2t})$. This means that $\frac{dC_{1t}}{C_{1t}} = \frac{dC_{2t}}{C_{2t}}$. The only difference in the consumption streams is then given by its initial value. These initial values are mainly determinated by the agent's wealth at the IPO time, specifically by $\omega_\tau$. Let us shortly analyse the behaviour of this fraction for different IPO triggers. Figure 4.6 shows the wealth fraction $\omega_\tau$ as function of the optimal market share in which the entrepreneur IPOs. Because of the dynamics of this new market in equilibrium, this fraction gets fixed at the IPO, and remains constant afterwards. Although most optimal IPO triggers occur at small market shares, we plot this fraction against almost all $s$ domain to illustrate the $S$ shape adopted for different values of risk aversion. For higher risk averse agents, this $S$ shape is steeper, and this shape tend to linearise as $\gamma$ approaches $0$. This effect is a direct result of the nonlinear structure of our pricing kernel described in (3.3). Today's wealth must equal the discounted consumption streams using the corresponding pricing kernel, as stated in (3.10).

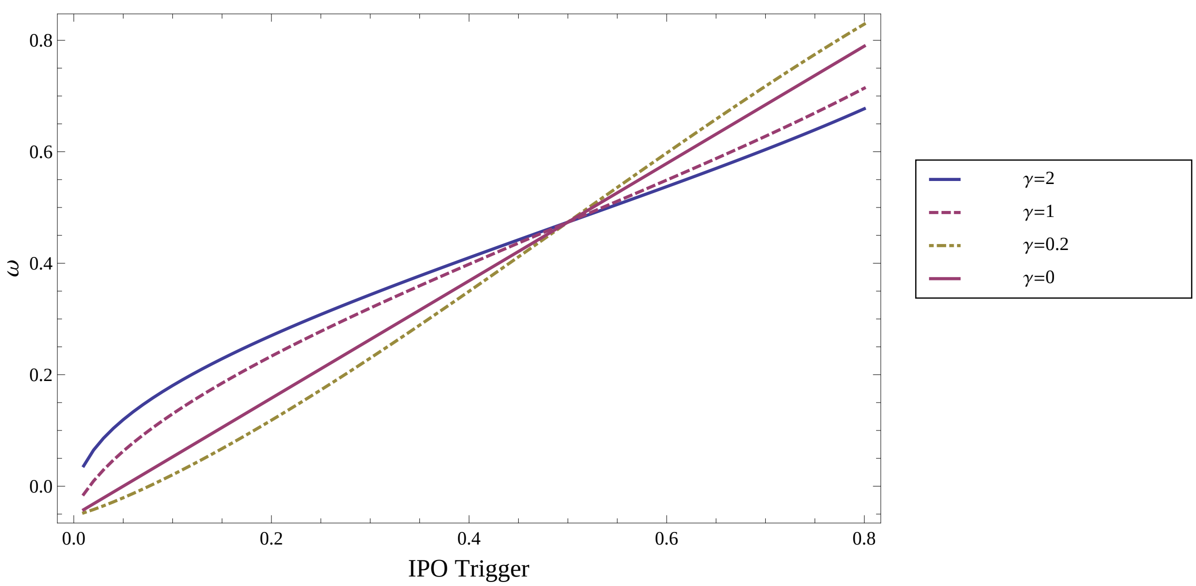



FIGURE 4.6. Aggregate consumption fraction consumed by entrepreneur for different IPO triggers. We plot for different values of risk aversion. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$, $\rho = 0$.

Once the initial consumption is determined by the wealth share of each agent, the process followed has the same structure for both, entrepreneur and market. As shown by Cochrane et al. (2007) and Martin (2007), the effect of portfolio diversification in the consumption processes is significant. Both consumption drift and diffusion are functions of

$s$. The consumption volatility is convex, and achieves its minimum for intermediate values of $s$, when the whole tradable market is not concentrated in a single asset. The maximum values of volatility is achieved in $s = 1$ or $s = 0$, when all the markets eggs are in a single basket. We resume these results in the following equations derived in Cochrane et al. (2007):

$$\frac{dC_{it}}{C_{it}} = [s_t \mu_{D_1} + (1 - s_t)\mu_{D_2}]\, dt + s_t \sigma_1 dZ_t + (1 - s_t)\sigma_2 dZ_t \tag{4.4}$$

Then

$$\mathbb{E}_t \left[\frac{dC_{it}}{C_{it}}\right] = [s_t \mu_{D_1} + (1 - s_t)\mu_{D_2}]\, dt \tag{4.5a}$$

$$\mathbb{V}ar_t \left[\frac{dC_{it}}{C_{it}}\right] = \left[s_t^2 \sigma_1 \sigma_1^T + (1 - s_t)^2 \sigma_2 \sigma_2^T + s_t(1 - s_t)\sigma_1 \sigma_2^T\right] dt \tag{4.5b}$$

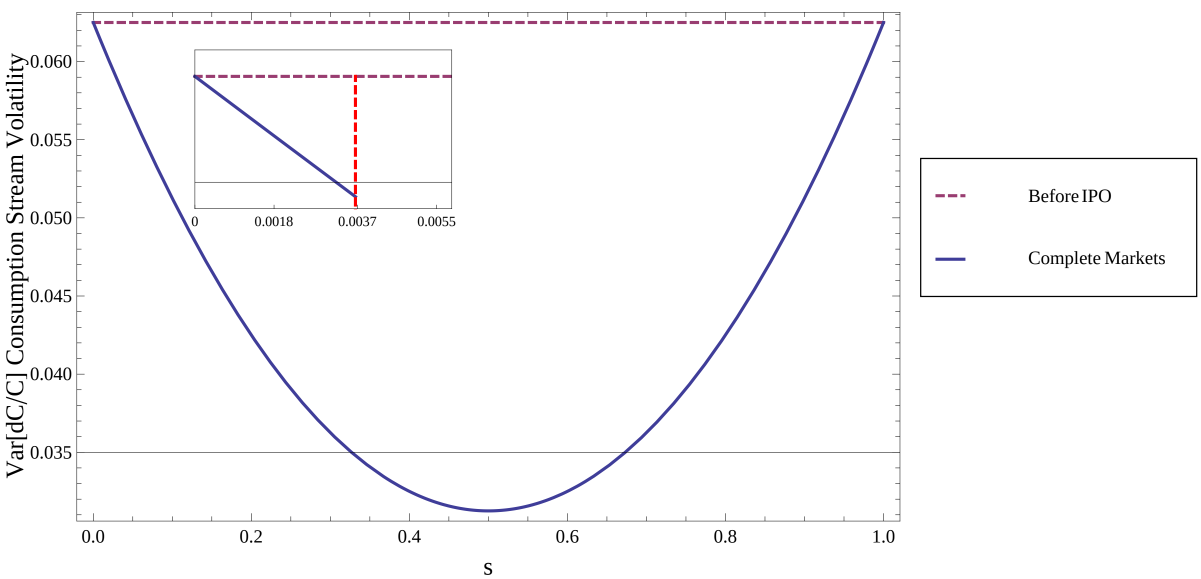



FIGURE 4.7. Consumption volatility as a function of the relative dividend. The dashed line shows the consumption volatility before IPO, the solid line the consumption volatility after IPO and the red dashed line shows the optimal IPO trigger. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$, $\rho = 0$ and $\gamma = 2$.

This means that the IPO diversification effect decreases both agents consumption stream volatility, improving the general market scenario. Figure(4.7) shows how the consumption process changes before versus after the IPO. Before the IPO, the agent's consumption volatility is the same as the dividends volatility, represented by the constant dashed line. After the IPO is triggered, agent's wealth is given by a fraction of the economy's aggregate wealth, and so the consumption rate stream is the same as the aggregate consumption rate stream: For intermediate values of $s$, volatility reaches its minimum. At the IPO time, consumption volatility falls enough to compensate for the fixed transaction costs.

As is well known from consumption-based models, a fall in consumption volatility reduces the need for precautionary savings. When endowments are more stable, the entrepreneur maximizes his utility consuming a larger amount of his wealth today. We will use the indirect utility approach derived in (3.22) to study the behaviour of entrepreneur's wealth related to his consumption.

Before the IPO, the entrepreneur perceives a wealth given by $W_{1t} = P_t^* - \alpha W_t$, where $P_t^*$ is the price obtained through the indirect utility approach, i.e. the price perceived of this asset plus the IPO timing option. His initial consumption is given by the company's dividend streams. After the IPO, entrepreneur's wealth is given by (3.14), and his consumption the same as in (2.8). Notice that after the IPO both agents have the same consumption-wealth ratio, given by $\frac{D_{1t}+D_{2t}}{W_{1t}+W_{2t}}$.

Figure 4.8 shows consumption-wealth ratio for the entrepreneur. Before the IPO, he consumes a lower fraction of his wealth comparing to the diversified scenario. After the IPO, marked with the red dashed line, entrepreneur's consumption jumps to the thick line. If we recall Figure 4.5, at the IPO time $P_\tau^* = P_\tau$, which means that the perceived interchangeable wealth of the entrepreneur does not change with the IPO. What does change is the consumption stream from $C_{1t} = D_{1t}$ to $C_{1t} = \omega\,(1-\alpha)\,(D_{1t} + D_{2t})$. This change produces an increase in consumption, as shown in previous figure. This implies that the entrepreneur is willing to consume a larger amount of his total wealth after the IPO, mainly explained by a reduction in his future endowments volatility. A decrease in the consumption volatility due to the IPO decision reduces the need for precautionary savings, intensifying the economy agent's consumption.

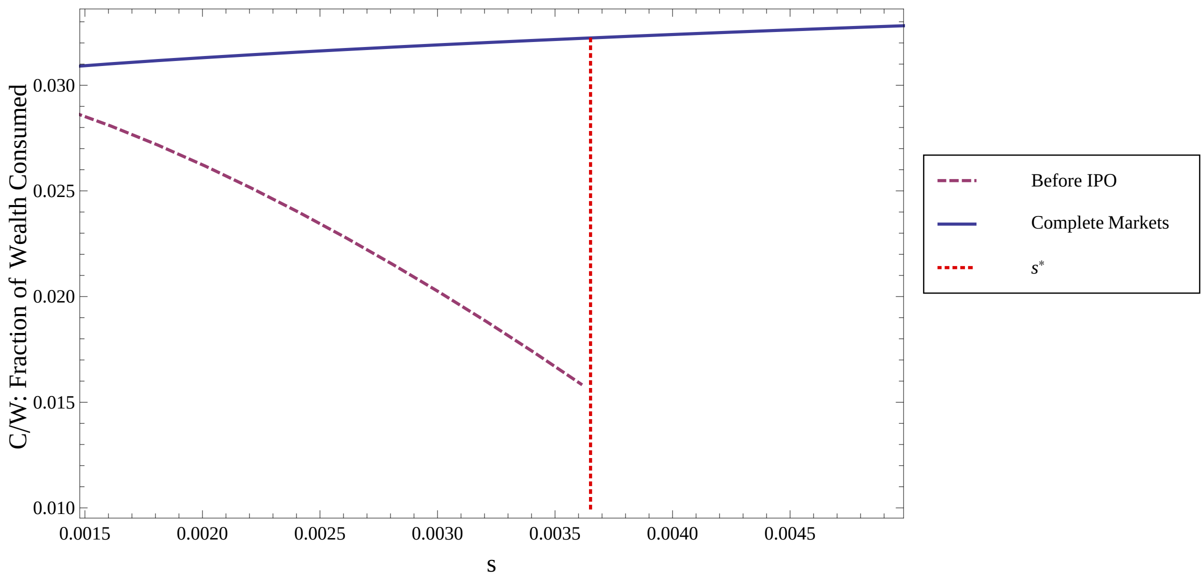



FIGURE 4.8. Consumption/wealth ratio for the entrepreneur as a function of the market share. The dashed line shows the consumption/wealth ratio before IPO, the solid line the ratio after IPO and the red dashed line shows the optimal IPO trigger. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$, $\rho = 0$ and $\gamma = 2$.

### 4.4. Private and Aggregate Economy's Welfare

Until now, we have analyse and studied the entrepreneur's optimal IPO trigger and it's diversification effect over agent's consumption. These analysis generates a natural question: Does the entrepreneur's optimal IPO timing is optimal for the economy as a whole? The answer to this question should be no. During the calculation of the optimal IPO trigger we where only seeking the maximization of the entrepreneur utility function. The decision of when to IPO was determined exclusively by the entrepreneur's welfare, without considering the aggregate utility of the economy. If we consider the benefits experienced by the rest of the market, the optimal market share to IPO may decrease. To determine the optimal IPO trigger that maximizes the aggregate economy, we should consider the aggregate utility of the economy, i.e.

$$V^{\tau}(D_{1t}, D_{2t}) = V^{1}(D_{1t}, D_{2t}) + V^{2}(D_{1t}, D_{2t}) \tag{4.6}$$

$$= \mathbb{E}_t \left[ \int_t^{\infty} e^{-\delta(s-t)} \left( u(C_{1s}) + u(C_{2s}) \right) \, ds \right] \tag{4.7}$$

$$= \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \left( u(D_{1s}) + u(D_{2s}) \right) \, ds + e^{-\delta(\tau-t)} \int_{\tau}^{\infty} e^{-\delta(s-\tau)} \left( u(C_{1s}) + u(C_{2s}) \right) \, ds \right] \tag{4.8}$$

The procedure to solve this optimization problem is the same as we did for the entrepreneur in Proposition 2.1 and Corollary 2.1. Following the same procedure used in this Proposition, and using the result (A.16) obtained in the Appendix, we see that the optimal IPO triggers solve the following equation:

$$\begin{aligned} & s_i^*(1-s_i^*) \left( j_S(s_i^*, \omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{\gamma/2}}{\delta - c\,(1-\gamma, 0)} - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{1-\gamma/2}}{\delta - c\,(0, 1-\gamma)} \right)' \\ & + \lambda_i \left( j_S(s_i^*, \omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{\gamma/2}}{\delta - c\,(1-\gamma, 0)} - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{1-\gamma/2}}{\delta - c\,(0, 1-\gamma)} \right) = 0 \\ & i = 1, 2 \end{aligned} \tag{4.9}$$

Where $j_S$ is the same as in (A.17), and $\lambda_i$ solves equation (A.13).

We test this new IPO triggers and compare them to the ones obtained before. In figure (4.9) we plot this new optimal triggers against the entrepreneur's optimal triggers, for different transactional costs. We notice that the overall economy welfare is maximised when the entrepreneur IPOs at smaller market shares. The utility lost by the entrepreneur when he IPOs at a lower share is compensated by diversification gains in the whole market.

The cost to reach this optimal in the overall economy can be measured by tracing horizontal lines in this figure, picking the optimal market share for a certain transactional cost, and measuring how much the transactional costs should be lowered so that the entrepreneur

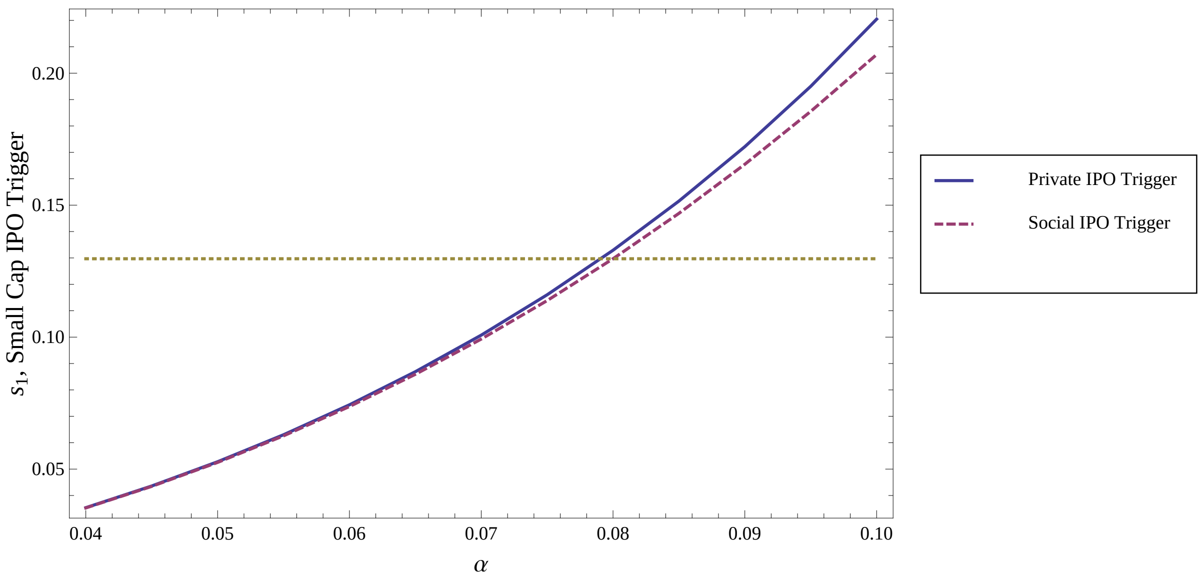



FIGURE 4.9. Aggregate economy and private small-cap IPO trigger against different transactional costs. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$ and $\gamma = 1$

IPOs at this share. As an example, suppose transactional costs are $\alpha = 8\%$. Under this case, the optimal trigger is $s^* = 0.129693$. If we look at the entrepreneur's curve following the horizontal line traced in the figure, this value is reached when $\alpha$ is lower, around $7,9\%$. This $0.1\%$ could be subsidise by an external entity, in order to reach the optimal value that maximises the aggregate economy welfare.

In a close exchange economy, in order to subsidise the entrepreneur we must tax the market. How much should we tax the market and subsidise the entrepreneur agent in order that the IPO timing is optimal for both? This decision has some differences from the one above, where the market never perceives any transactional costs. Suppose the market pays now a fraction $\alpha_M < \alpha$ of the transactional costs. Some equilibrium conditions change under this scenario. Under analogous procedures, agents' wealth at the IPO is given by

$$W_{1\tau} = P_{1\tau} - (\alpha - \alpha_M)(P_{1\tau} + P_{2\tau}) \quad (4.10)$$

$$W_{2\tau} = P_{2\tau} - \alpha_M (P_{1\tau} + P_{2\tau}) \quad (4.11)$$

$$(4.12)$$

Which means the fraction consumed $\omega_\tau$ is now

$$\omega_\tau = \frac{P_{1\tau} - (\alpha - \alpha_M)(P_{1\tau} + P_{2\tau})}{(1-\alpha)(P_{1\tau} + P_{2\tau})} \tag{4.13}$$

Now that the market perceives a fraction of the transactional costs, there's an optimal IPO share that maximises his utility. This optimal timing problem is solved using proposition 2.1, properly changing $j(s, \omega_\tau)$ for $j_M(s, \omega_\tau)$ calculated in (A.20). The optimal IPO trigger for the market agent is the market share $s*$ that solves

$$\begin{aligned} & s_i^*(1-s_i^*)\left(j_M(s_i^*, \omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{1-\gamma/2}}{\delta - c(0, 1-\gamma)}\right)' \\ & + \lambda_i \left(j_M(s_i^*, \omega(s_i^*)) - \frac{\left(\frac{1-s_i^*}{s_i^*}\right)^{1-\gamma/2}}{\delta - c(0, 1-\gamma)}\right) = 0 \\ & i = 1, 2 \end{aligned} \tag{4.14}$$

With $\lambda_i$ solving (A.13). The entrepreneur also IPOs at a lower level, now that his transactional costs are lower. This IPO trigger is obtained solving the equations from corollary 2.1 adjusting our new $\omega_\tau$. It can easily be proved that this new optimal IPO market share represents the IPO trigger that maximises the aggregate economy utility:

$j_S(s, \omega_\tau) = j(s, \omega_\tau) + j_M(s, \omega_\tau)$, therefore if $s^*$ solves corollary 2.1 and equation (4.14), it will also solve our aggregate economy optimal condition (4.9).

Thus, we should solve both optimal timing problems and seek the tax/subsidy that equals the optimal IPO shares. Continuing with our previous example, we fix $\alpha = 8\%$, and solve our optimal problems for different values of $\alpha_M$. Figure 4.10 shows this results. We see that as the subsidy increases, the market share at which the entrepreneur IPOs decreases. The entrepreneur perceives lower transactional costs, and is willing to IPO at a lower share. On the other hand, the optimal share to IPO increases from the market point of view, as he is increasingly paying a greater fraction of the IPO transactional costs. The

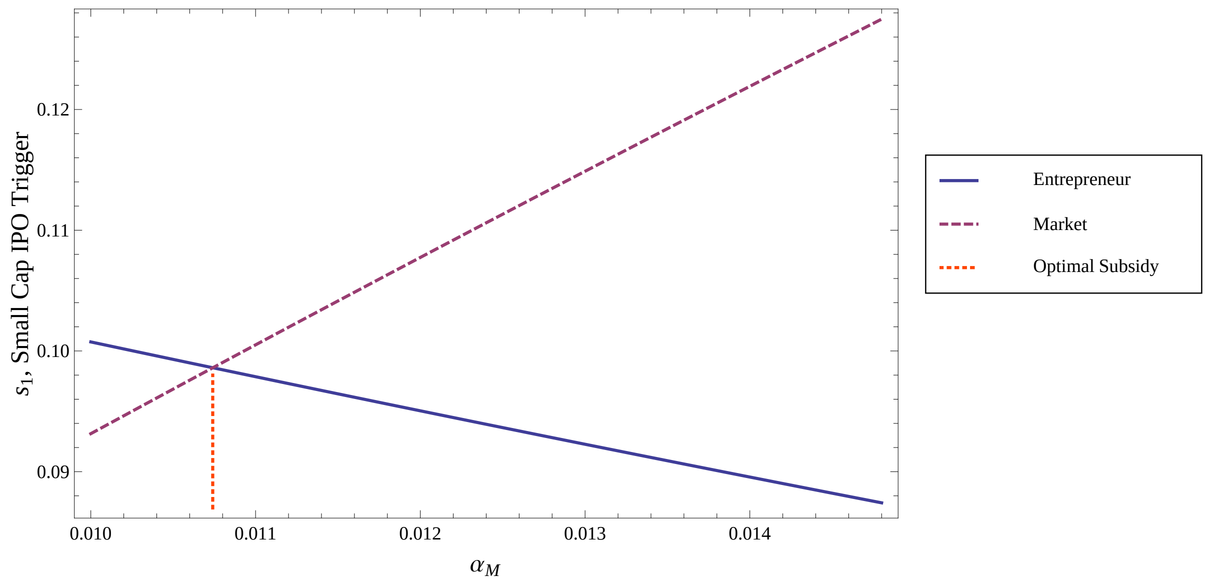



FIGURE 4.10. Aggregate economy and private small-cap IPO trigger against different transactional costs. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.08$ and $\gamma = 1$

lines cross around $\alpha_E = 1,074\%$, which represent the optimal subsidy/tax that maximises the economy aggregate utility. Under this structure, the market should pay $\alpha_E$, reducing IPO's costs to $\alpha - \alpha_E$, which leads the entrepreneur to IPO at a level that is optimal for both. The early diversified scenario is enough to compensate the costs paid by the market agent.

### 4.5. Relation Between Betas and IPO Triggers

The following section studies the betas of firms at the optimal IPO time. We wish to analyse the relation between the size at which a firm IPOs and its Betas. This way we may measure the diversification effect over returns of assets. For conducting our purpose, we define the Beta in terms of excess returns from the CAPM [1], i.e.

$$\mathbb{E}_t\left[r_{1t}\right] - r_t^f = \beta\left(\mathbb{E}_t\left[r_{Mt}\right] - r_t^f\right) \tag{4.15}$$

[1]Note that the CAPM is valid because at the IPO the asset is part of the complete market.

Where $\mathbb{E}_t[r_{1t}]$ is obtained from (3.18), $r_t^f$ as in (3.16), and $r_{Mt}$ is the market return after the IPO, given by the weighted sum of both asset's returns. $\beta$ is isolated from this equation.

We wish to consider companies from several industries. To keep industries comparable, dividend processes will have the same drift and volatility. Thus, the correlation of its dividends with the market is what will distinguish one industry from another. Different industries will be characterized by the correlation coefficient, and different companies within an industry will be characterized by their market share. This way, within an industry we may have many companies of different sizes, but whose dividend streams follow the same process.

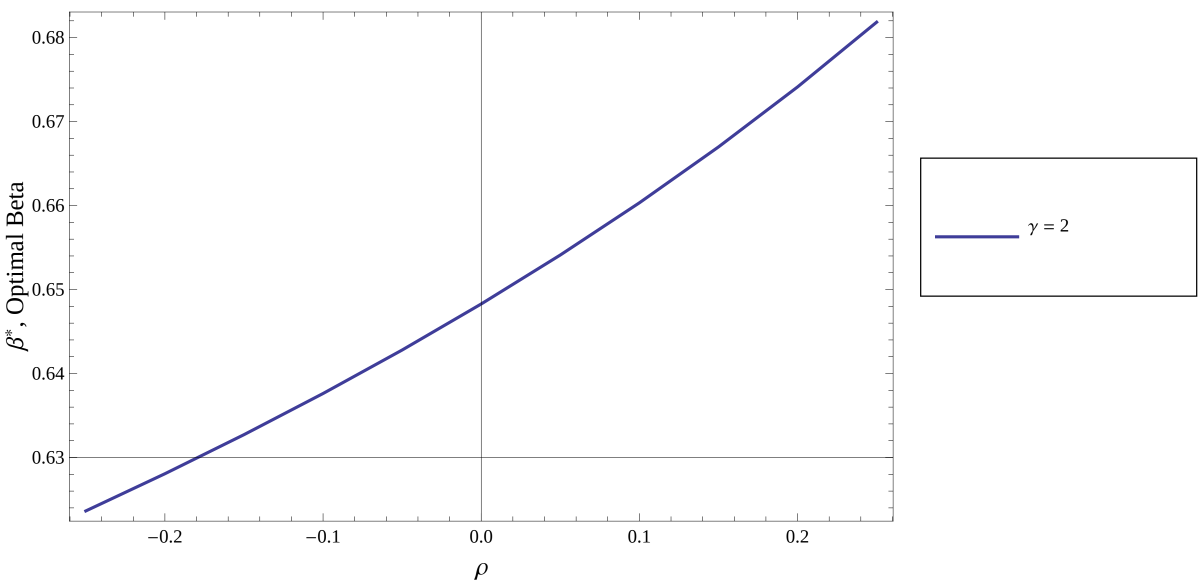



FIGURE 4.11. Beta of firm 1 at optimal IPO time against correlation between dividend payoffs. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25\sqrt{1-\rho^2}$, $\sigma_{12} = 0.25\rho$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$ and $\gamma = 2$.

Figure 4.11 shows the relation of the Beta at which a company optimally triggers the IPO against correlation of the dividend streams. We see that the higher the correlation of the industry with the market, the higher the Beta at which the company triggers the IPO. This information can be added to information from Figure 4.1, and we can obtain the relation showed in next figure.

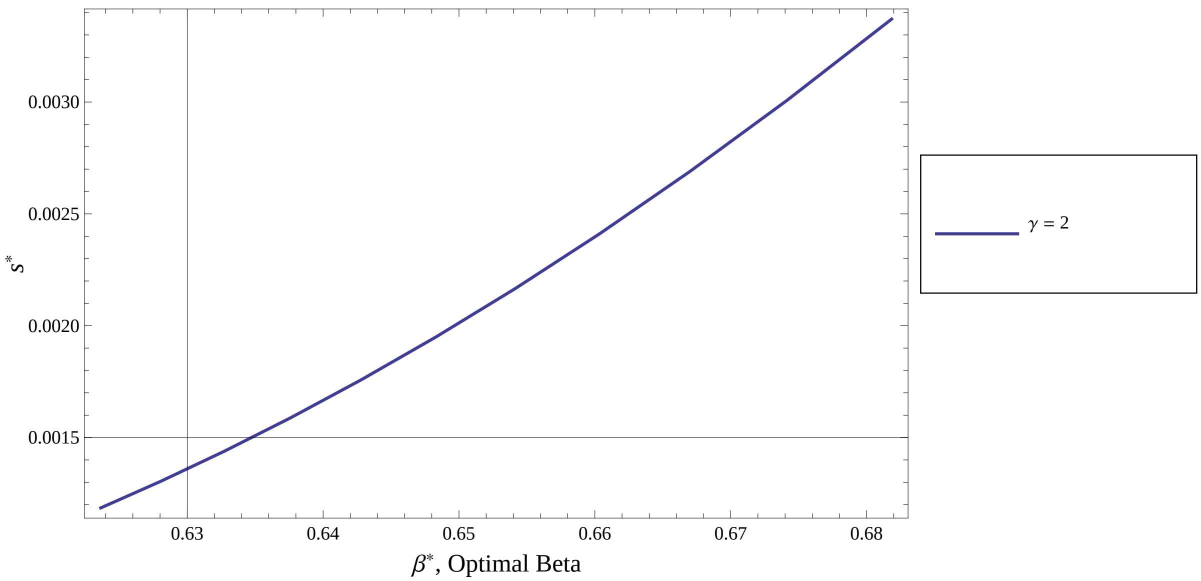



FIGURE 4.12. Small-cap trigger against the beta of firm 1. Both variables are monotonic functions of the correlation $\rho$. The parameters used are $\mu_{D_1} = \mu_{D_2} = 0.05$, $\sigma_{11} = 0.25\sqrt{1-\rho^2}$, $\sigma_{12} = 0.25\rho$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\alpha = 0.05$ and $\gamma = 2$.

Figure 4.12 shows the relations between the optimal size at which a company IPOs versus its Beta in that moment. Industries with lower correlation with the market IPOs with lower Betas, and companies within this industries will IPO with lower sizes. We observe that there is a direct relation between optimal $s$ triggers and Betas. Considering that Beta measures the degree at which risk premium follow the market's risk premium, industries with lower covariance with the market should IPO with lower market shares. This kind of industries experience more diversification benefits with an IPO.

The model predicts an economy evolution over time. Remember Figure 4.3: The smaller the IPO trigger, the lower is the expected time a company has to wait to IPO. Because industries that IPO with lower betas has smaller optimal triggers, the first companies to IPO are expected to come from this industries. This predictions were captured empirically in the work of Astudillo (2008), where he compares, via lineal regressions, countries Beta from economies in different stages of development. He concludes that at early stages of the economy the first firms to IPO comes from industries that show lower betas, overweighting these sectors in the market. As the economy develops, firms from higher Beta industries IPO, raising the market's Beta, until the market reaches a mature point in which al sectors in the economy are well represented.

### 4.6. Dividends subject to jumps: Idiosyncratic and Systematic Shocks

We will now analyse the effect of dividend jumps in the IPO decision. We will take into account 4 scenarios: No jumps, Positive Jumps, Negative Jumps, and Economical Crisis. The scenario with positive jumps may represent a developing industry in which unexpected technological advances strongly increases dividend streams. The case with negative jumps can represent an industry exposed to unexpected disasters, say droughts in agriculture industries. An economical crisis is referred to an unexpected negative shock affecting both, industry and whole market. We compensate the drift in each case so that the scenarios al comparable. In Positive Jumps we consider normal jumps affecting the only the company, with mean $0.05$, variance $0.1^2$ and an occurrence rate of $1/5$ (1 every 5 years). Negative Jumps are the same, but with mean $-0.05$. Economical Crisis are characterized by jumps affecting both assets (Company and Market) with mean $-0.1$, variance $0.1^2$ and occurrence rate $1/20$.

Figure 4.13 shows how the IPO trigger is affected by jump scenarios. When correlation of dividends is high, we see that in the case of jumps affecting only the unlisted company IPO triggers are lower. The presence of jumps increases the volatility of endowments, increasing diversification benefits of a diversified portfolio. An entrepreneur in an industry which suffers unexpected jumps is better selling his company at a lower market share and acquiring a fraction of the new market portfolio instead. The opposite effect is observed in an scenario of economical crisis when dividends are highly correlated. In this scenario, jumps not only affect the asset of the entrepreneur, but also the whole market. Diversification benefits decrease in the presence of this systematic shock on the whole market portfolio, and an IPO is less desirable. The entrepreneur agent waits for higher market shares to compensate the acquisition of this less diversified market portfolio.

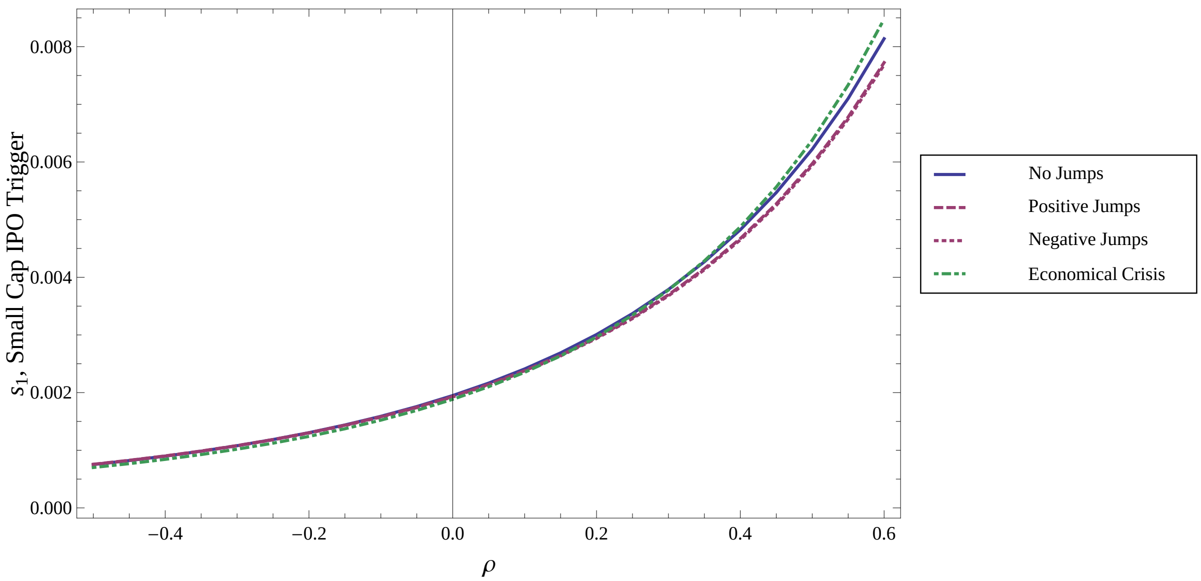



FIGURE 4.13. Small-cap IPO trigger with respect to the correlation between both dividend payoffs, $\rho$, when we allow for jumps in the dividend processes. The parameters used are $\mu_{D_i} = 0.05 - w_i * \left(e^{\mu_i^J + \frac{(\sigma_i^J)^2}{2}}\right)$, $\sigma_{11} = 0.25\sqrt{1-\rho^2}$, $\sigma_{12} = 0.25\rho$, $\sigma_{21} = 0$, $\sigma_{22} = 0.25$, $\delta = 0.04$, $\gamma = 2$, and $\alpha = 0.05$, where $w_i$, $\mu_i^J$ and $\sigma_i^J$ represent the frequency, size mean, and size variance of the jumps in each scenario

# 5. CONCLUSION AND FUTURE RESEARCH

We summarize the basic procedure conducted in this thesis. First we built up a general equilibrium model in an endowment economy to analyse an IPO scenario. The model is solved using dynamic programming and asset pricing results from the works of Cochrane et al. (2007); Martin (2007). We analysed the random variables and states affecting the decision, studying expected optimal hitting times and diversification effects over the entrepreneur and the market.

Eliminating any extra benefit, portfolio diversification is strong enough to trigger an IPO: Consumption streams are less volatile, which increases entrepreneur's utility through time, overcoming transactional costs. Within an industry, the relative size of the firm to the market is the main characteristic that triggers the IPO. As a company within an industry grows, IPO diversification benefits increases. This leads to a more diversified market portfolio after the IPO, which is highly desirable for risk averse entrepreneur and market.

Studying expected hitting times, the model predicts that within an industry bigger firms are expected to IPO first (greater diversification benefits). In our model, each industry defines a dividend stream, and each dividend stream generates an optimal IPO trigger. As a firm approaches this dividend share, the time expected to IPO approaches to zero.

Before the IPO, we can measure the costs of turning an illiquid asset tradable as the difference between the price at which the entrepreneur IPOs and its actual market value. The entrepreneur will not IPO unless IPO incomes generate sufficient diversification benefits. As the asset's market share approaches the IPO trigger, the costs of turning liquid the asset decrease towards zero, until the entrepreneur is willing to IPO at the market value of his firm.

Optimal IPO triggers are higher when dividends face high correlation with the market, an so are the CAPM Betas in the moment of the IPO. Merging this two effects, we realize that IPO trigger and the Beta of the newly public firms move in the same direction. This means (considering that the lower IPO trigger the lower the expected time to IPO) that the first firms expected to IPO should have lower Betas. This effect might explain why most public firms in developing economies and emerging markets are concentrated in low Betas sectors. As time passes, the model predicts that firms with higher Betas will be incorporated in the market until the market reaches a maturity point.

The IPO trigger that maximises the aggregate economy welfare is lower than the one that maximises entrepreneur's utility. Comparing this two optimal IPO triggers for different transactional costs, we can evaluate how much it costs to the economy this closed IPO structure. This costs may disappear if part of the transactional costs are shared by the market agent. This way the costs to IPO are lowered and the entrepreneur may IPO at a lower share, enough to compensate the subsidy paid by the market. The existence of the taxes and subsidies in an IPO is optimal for the aggregate economy.

It is left for future research improvements to the model that could generate further analysis. It will probably require the incorporation of more trees in the economy (exploiting the results from Martin (2007)), which introduces more the degrees of freedom to the model. With more assets and representative agents in the market, we could be able to measure prices, returns, and covariance structures before and after the IPO. We could then compare it with empirical evidence (Braun & Larrain (2009)) and test the results of this model. We could also fit the dividend streams (i.e. drift, diffusion, and jumps) to the data and introduce more specific transactional costs. For some analysis, we could desire a more complex capital budgeting structure (as seen on Chen et al. (2009)): Allow the entrepreneur the use of leverage, keep part of his capital in a liquid diversified portfolio, and default on debt. The model, beside complementing IPO theory, might be used as a guide for any entrepreneur analysing IPO timing decision, considering portfolio diversification benefits, and public entities that wish to maximise the positive diversification effects over the aggregate economy.

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## APPENDIX A. OPTIMAL STOPPING CONTROL

### A.1. Obtaining an expression for J

$$j_E(\tau, D_{1t}, D_{2t}) = e^{-\delta(\tau-t)}\mathbb{E}_\tau\left[\int_\tau^\infty e^{-\delta(s-\tau)}\left(\frac{C_{1s}^{1-\gamma}-1}{1-\gamma} - \frac{D_{1s}^{1-\gamma}-1}{1-\gamma}\right)ds\right]$$
$$= \frac{e^{-\delta(\tau-t)}}{1-\gamma}\left(\underbrace{\int_\tau^\infty e^{-\delta(s-\tau)}\mathbb{E}_t\left[C_{1s}^{1-\gamma}\right]ds}_{(Et1)} - \underbrace{\int_\tau^\infty e^{-\delta(s-\tau)}\mathbb{E}_\tau\left[(D_{1s})^{1-\gamma}\right]ds}_{(Et2)}\right) \tag{A.1}$$

We can notice from equations (3.2) and (3.10) that $(Et1) = W_{1\tau}/C_{1\tau}^{\gamma}$. Using the expressions (3.14) for $W_{1t}$ and (3.6) for the prices, and noticing that $C_{1t} = \omega_\tau\left(D_{1t}+D_{2t}\right)(1-\alpha)$

$$(Et1) = D_{1\tau}\left(D_{1\tau}D_{2\tau}\right)^{-\gamma/2} j(s,\omega_\tau) \tag{A.2}$$

With

$$j(s,\omega_\tau) = W_{1\tau}/C_{1\tau}^{\gamma} \tag{A.3}$$
$$= \left(\omega_\tau\left(1-\alpha\right)\right)^{1-\gamma}\int_{-\infty}^{\infty}\left(\frac{1-s_\tau}{s_\tau}\right)^{iv}\left(\frac{1}{\delta-c_1(v)} + \frac{(1-s_\tau)/s_\tau}{\delta-c_2(v)}\right)\Psi_\gamma(v)\,dv \tag{A.4}$$

$(Et2)$ is obtained after solving the conditional expectation:

$$\mathbb{E}_t\left[D_{1s}^{1-\gamma}\right] = \mathbb{E}_\tau\left[e^{(1-\gamma)(y_{1\tau}+\tilde{y}_{1(s-\tau)})}\right] = e^{(1-\gamma)y_{1t}}e^{(s-\tau)\mathbf{c}((1-\gamma),0)} \tag{A.5}$$

Then

$$(Et2) = \frac{D_{1\tau}^{1-\gamma}}{\delta-\mathbf{c}((1-\gamma),0)} = \frac{D_{1\tau}\left(D_{1\tau}D_{2\tau}\right)^{-\gamma/2}\left(\frac{1-s_\tau}{s_\tau}\right)^{\gamma/2}}{\delta-\mathbf{c}((1-\gamma),0)} \tag{A.6}$$

for $\delta_1 - \mathbf{c}((\gamma-1),0) > 0$. Finally

$$j_E(\tau, D_{1t}, D_{2t}) = \frac{e^{-\delta(\tau-t)}D_{1\tau}\left(D_{1\tau}D_{2\tau}\right)^{-\gamma/2}}{1-\gamma}\left(j(s,\omega_\tau) - \frac{\left(\frac{1-s_\tau}{s_\tau}\right)^{\gamma/2}}{\delta - \mathbf{c}((\mathbf{1}-\gamma),\mathbf{0})}\right) \quad \text{(A.7)}$$

### A.2. Optimal Stopping States

At first, our state variables are the dividend processes $D_{1t}$ and $D_{2t}$ given by the following PDE:

$$\frac{dD_{it}}{D_{it}} = \mu_{D_i}dt + \sigma_i dZ_t + \sum_{k=1}^{l}\int_{\mathbb{R}}\left(e^{g_{ik}(z)} - 1\right)N^{(k)}(dt,dz) \quad \text{(A.8)}$$

The optimal control problem consists in finding the optimal states in which $J^\tau(t, D_{1t}, D_{2t})$ reaches its maximum value, or equivalently, the optimal states in which $j_E(\tau, D_{1t}, D_{2t})$ is maximized. The problem is summarized in equations (2.12). We apply Theorem (2.2) from Øksendal & Sulem (2005) for finding the value function $V$:

Define the 3-dimensional state process

$$Z_s = \begin{bmatrix} t+s \\ D_{1s} \\ D_{2s} \end{bmatrix}, \qquad Y_0 = \begin{bmatrix} t \\ x \\ y \end{bmatrix} \quad \text{(A.9)}$$

Define also $\widehat{S} = \mathbb{R}^3_+$, $\widehat{D} = \{z \in \widehat{S} \backslash \phi(z) > j_E(z)\}$, $\tau_{\widehat{S}} = \inf\{t \in \mathbb{R}^+ \backslash Z_t \notin \widehat{S}\}$ and $\mathcal{T} = \{\tau \text{ stopping time } \backslash \tau < \tau_{\widehat{S}}\}$.

We must find $\phi : \overline{S} \to \mathbb{R}$ that satisfies the following

(i) $\partial D$ is a Lipschitz surface

(ii) $\phi \geq j_E$ on S

(iii) $\phi \in C^1(\widehat{S}) \cup C(\overline{\widehat{S}})$ and $\phi \in C^2(\widehat{S} \setminus \partial D)$ with locally bounded derivatives near $\partial D$

(iv) $\mathbb{E}\left[\int_t^{\tau_{\widehat{S}}} \mathbb{1}_{\partial D}(Y_s)\,ds\right] = 0$

(v) $Y_{\tau_{\widehat{S}}} \in \partial\widehat{S}$ a.s. on $\{\tau_{\widehat{S}} < \infty\}$ and $\lim_{s\to\tau_{\widehat{S}}^-} \phi(Y_s) = j_E(Y_{\tau_{\widehat{S}}})\mathbb{1}_{\{\tau_{\widehat{S}}<\infty\}}$

(vi) $\mathbb{E}\left[|\phi(Y_\tau)| + \int_t^{\tau_{\widehat{S}}} \left(|A\phi(Y_s)| + \|\sigma^T \nabla\phi(Y_s)\|^2 + \sum_{k=1}^{l} |\phi(Y_t e^{g_k(z_k)}) - \phi(Y_t)|^2 \, d\nu_k(z_k)\right) ds\right] < \infty \; \forall\, \tau \in \mathcal{T}$

(vii) $A\phi = 0$ on $D$

(viii) $A\phi \leq 0$ on $S \setminus \partial D$

(ix) $\tau_D := \inf\{t \in \mathbb{R} \backslash Z_t \notin D\} < \infty$ a.s.

(x) $\{\phi(Y_\tau) \backslash \tau \in \mathcal{T}\}$ is uniformly integrable, for all $(t, x, y) \in \mathbb{R}^3_+$

We begin searching for a function $\phi(t, x, y)$ that fits the conditions above. The generator of the process $Z_t$ is given by

$$A\phi(t,x,y) = \frac{\partial \phi}{\partial t} + x\mu_{D_1}\frac{\partial \phi}{\partial x} + y\mu_{D_2}\frac{\partial \phi}{\partial y} + \frac{1}{2}A_1A_1^T x^2 \frac{\partial^2 \phi}{\partial x^2} + A_1A_2^T xy\frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{2}A_2A_2^T y^2\frac{\partial^2 \phi}{\partial y^2} + \sum_{k=1}^{l}\int_{\mathbb{R}}\Bigg(\phi(t, x + x(e^{g_{1k}(z)} - 1), y + y(e^{g_{2k}(z)} - 1)) - \phi - \frac{\partial \phi}{\partial x}x(e^{g_{1k}(z)} - 1) - \frac{\partial \phi}{\partial y}y(e^{g_{2k}(z)} - 1)\Bigg)\nu^{(k)}(dz) \tag{A.10}$$

Condition (vii) requires that $A\phi = 0$. We suppose that $\phi(t, x, y)$ has the following structure:$\phi(t, x, y) = e^{-\delta_1 t}x(xy)^{-\gamma/2}h(u)$, with $u$ given by $u_t \equiv \log\frac{1-s}{s} = \log y/x$. Replacing in equation above:

$$A\phi(t,x,y) = e^{-\delta_1 t}x(xy)^{-\gamma/2}\Bigg(\theta_0 h(u) + \theta_1 h'(u) + \theta_2 h''(u) + \sum_{k=1}^{l}\int_{\mathbb{R}}\Bigg(e^{g_{1k}(z)}e^{-\gamma/2(g_{1k}(z_k)+g_{2k}(z))}h(u + g_{2k}(z) - g_{1k}(z)) + h(u)\left[e^{g_{1k}(z)}(\gamma/2 - 1) + e^{g_{2k}(z)}\gamma/2 - \gamma\right] + h'(u)\left[e^{g_{1k}(z)} - e^{g_{2k}(z)}\right]\Bigg)d\nu^{(k)}(z)\Bigg) \tag{A.11}$$

where

$$\theta_0 = -\delta + (1-\gamma/2)\mu_{D_1} - \gamma/2\mu_{D_2} + 1/2(\gamma^2/4 - \gamma/2)A_1A_1^T$$
$$+ (\gamma^2/4 - \gamma/2)A_1A_2^T + 1/2(\gamma^2/4 + \gamma/2)A_2A_2^T \quad \text{(A.12)}$$
$$\theta_1 = \mu_{D_2} - \mu_{D_1} + 1/2(\gamma - 1)A_1A_1^T + A_1A_2^T + 1/2(-\gamma - 1)A_2A_2^T$$
$$\theta_2 = 1/2A_1A_1^T - A_1A_2^T + 1/2A_2A_2^T$$

We propose $h(u) = e^{\lambda u}$:

$$A\phi(t,x,y) = e^{-\delta_1 t}x(xy)^{-\gamma/2}e^{\lambda u}\Bigg(\theta_0 + \theta_1\lambda + \theta_2\lambda^2 +$$
$$\sum_{k=1}^{l}\int_{\mathbb{R}}\left(e^{g_{1k}(z)(1-\lambda-\gamma/2)-g_{2k}(z)(-\lambda+\gamma/2)} + e^{g_{1k}(z)}(\gamma/2+\lambda-1) + e^{g_{2k}(z)}(\gamma/2-\lambda) - \gamma\right)d\nu^{(k)}(z)\Bigg)$$

This way, condition (vii) is satisfied when $\lambda$ solves

$$\theta_0 + \theta_1\lambda + \theta_2\lambda^2 +$$
$$\sum_{k=1}^{l}\int_{\mathbb{R}}\left(e^{g_{1k}(z)(1-\lambda-\gamma/2)-g_{2k}(z)(-\lambda+\gamma/2)} + e^{g_{1k}(z)}(\gamma/2+\lambda-1) + e^{g_{2k}(z)}(\gamma/2-\lambda) - \gamma\right)d\nu^{(k)}(z) = 0$$
(A.13)

which has two solutions in the real line: $\lambda_1$ and $\lambda_2$.

Conditions (i), (iii)-(iv), and (vii) are satisfied trivially by construction of the solution. (v) is a consequence of the construction of the solution and (i). We should now check (ii), (vi), (viii)-(x). These are not easy steps due to the structure of function $j$. During this work I just checked that for the parameters chosen the conditions where satisfied. It is left for future research to formally determine the condition over the parameters to ensure an optimal solution.

### A.3. Aggregate Economy and Market Agent

The procedure for solving the economy's optimal trigger is analogous as we just did for the entrepreneur: We regroup the terms that depend on $\tau$ in (4.6) so that maximizing $V^{\tau}$ is equivalent to maximize $\mathbb{E}_t\left[J_S(D_{1\tau}, D_{2\tau})\right]$, where

$$J_S(D_{1\tau}, D_{2\tau}) = e^{-\delta(\tau-t)}\mathbb{E}_{\tau}\left[\int_{\tau}^{\infty} e^{-\delta(s-\tau)}\left(u(C_{1s}) + u(C_{2s}) - u(D_{1s}) - u(D_{2s})\right)\, ds\right] \tag{A.14}$$

$$= \frac{e^{-\delta(s-\tau)}}{1-\gamma}\left(\frac{W_{1\tau}}{C_{1\tau}^{\gamma}} + \frac{W_{2\tau}}{C_{2\tau}^{\gamma}} - \frac{D_{1\tau}^{1-\gamma}}{\delta - c\,(1-\gamma, 0)} - \frac{D_{2\tau}^{1-\gamma}}{\delta - c\,(0, 1-\gamma)}\right) \tag{A.15}$$

This last expression can be written as

$$J_S(x,y) = \frac{e^{-\delta(\tau-t)}x\,(xy)^{-\gamma/2}}{1-\gamma}\left(j_S\,(s,\omega_{\tau}) - \frac{\left(\frac{1-s}{s}\right)^{-\gamma/2}}{\delta - \mathbf{c}\,(1-\gamma,0)} - \frac{\left(\frac{1-s}{s}\right)^{1-\gamma/2}}{\delta - \mathbf{c}\,(0,1-\gamma)}\right) \tag{A.16}$$

where

$$j_S\,(s,\omega_{\tau}) = (1-\alpha)^{1-\gamma}\left(\omega_{\tau}^{1-\gamma} + (1-\omega_{\tau})^{1-\gamma}\right)$$
$$\int_{-\infty}^{\infty}\left(\frac{1-s_{\tau}}{s_{\tau}}\right)^{iv}\left(\frac{1}{\delta - c_1(v)} + \frac{(1-s_{\tau})/s_{\tau}}{\delta - c_2(v)}\right)\Psi_{\gamma}(v)\, dv \tag{A.17}$$

This procedure is also valid for the market agent, maximising $\mathbb{E}_t\left[J_M(D_{1\tau}, D_{2\tau})\right]$, with

$$J_M(D_{1\tau}, D_{2\tau}) = e^{-\delta(\tau-t)}\mathbb{E}_{\tau}\left[\int_{\tau}^{\infty} e^{-\delta(s-\tau)}\left(u(C_{2s}) - u(D_{2s})\right)\, ds\right] \tag{A.18}$$

And solving this as we did before:

$$J_M(x,y) = \frac{e^{-\delta(\tau-t)}x\,(xy)^{-\gamma/2}}{1-\gamma}\left(j_M\,(s,\omega_{\tau}) - \frac{\left(\frac{1-s}{s}\right)^{1-\gamma/2}}{\delta - \mathbf{c}\,(0,1-\gamma)}\right) \tag{A.19}$$

with

$$j_M\left(s,\omega_\tau\right) = \left(1-\alpha\right)^{1-\gamma}\left(1-\omega_\tau\right)^{1-\gamma} \int_{-\infty}^{\infty}\left(\frac{1-s_\tau}{s_\tau}\right)^{iv}\left(\frac{1}{\delta-c_1(v)}+\frac{(1-s_\tau)/s_\tau}{\delta-c_2(v)}\right)\Psi_\gamma(v)\,dv \tag{A.20}$$

## APPENDIX B. CUMULANT-GENERATING FUNCTION

Cumulant-Generating function $\mathbf{c}(v)$ is defined as the logarithm of the moment-generating function

$$\mathbf{c}(v) := \log\left(\mathbb{E}\left[e^{\mathbf{v}'(\tilde{y_{t+1}}-\tilde{y_t})}\right]\right) \tag{B.1}$$

In our case

$$\mathbf{c}(\mathbf{v}) = \mathbf{v}'\mu + \frac{1}{2}\mathbf{v}'\Sigma\mathbf{v} + \omega\left(\mathbb{E}e^{\mathbf{v}'S_k} - 1\right) \tag{B.2}$$

$$\mathbf{c}(\mathbf{v}) = \mathbf{v}'\mu + \frac{1}{2}\mathbf{v}'\Sigma\mathbf{v} + \sum_{k=1}^{l}\int_{\mathbb{R}}\left(e^{\mathbf{v}'g^k(z_k)} - 1\right)\nu^{(k)}(dz_k)^1 \tag{B.3}$$

where $\Sigma = AA'$. If $S_k \sim N(\mu_S, \Sigma_S)$, $\mathbf{c}(v)$ is given by

$$\mathbf{c}(v) = \mathbf{v}'\mu + \frac{1}{2}\mathbf{v}'\Sigma\mathbf{v} + \omega\left(e^{\mathbf{v}'\mu_S + 1/2\mathbf{v}'\Sigma_S\mathbf{v}} - 1\right) \tag{B.4}$$

## APPENDIX C. PRICES IN EQUILIBRIUM

The method for obtaining asset prices in complete markets can be found in Martin (2007). We repeat the same procedure: It was previously stated that asset prices are given by (3.1) and (3.3). Using both, we get

$$P_{it} = \mathbb{E}_t \left[ \int_t^{\infty} e^{-\delta(s-t)} \left( \frac{D_{1s} + D_{2s}}{D_{1t} + D_{2t}} \right)^{-\gamma} D_{is} \, ds \right] \tag{C.1}$$

$$= \int_t^{\infty} e^{-\delta(s-t)} \left( D_{1t} + D_{2t} \right)^{\gamma} \mathbb{E}_t \left[ \frac{D_{is}}{(D_{1s} + D_{2s})^{\gamma}} \right] ds \tag{C.2}$$

For solving $\mathbb{E}_t \left[ \frac{D_{is}}{(D_{1s}+D_{2s})^{\gamma}} \right]$, we use the fact that

$$\mathbb{E}_t \left[ \frac{D_{is}}{(D_{1s} + D_{2s})^{\gamma}} \right] = e^{y_{it}} \mathbb{E}_t \left[ \frac{e^{\tilde{y}_{i(s-t)}}}{\left( e^{y_{1t}+\tilde{y}_{1(s-t)}} + e^{y_{2t}+\tilde{y}_{2(s-t)}} \right)^{\gamma}} \right]$$

$$= e^{y_{it}} e^{-\gamma/2(y_{1t}+y_{2t})} \mathbb{E}_t \left[ \frac{e^{(\mathbb{1}_{\{i=1\}}-\gamma/2)\tilde{y}_{1(s-t)}+(\mathbb{1}_{\{i=2\}}-\gamma/2)\tilde{y}_{2(s-t)}}}{\left( 2\cosh \frac{y_{2t}-y_{1t}+\tilde{y}_{2(s-t)}-\tilde{y}_{1(s-t)}}{2} \right)^{\gamma}} \right] \tag{C.3}$$

We now use de Fourier transform $\Psi_{\gamma}(v)$ of $1/[\cosh(u/2)]^{\gamma}$ for $\gamma > 0$ given by

$$\frac{1}{[2\cosh(u/2)]^{\gamma}} = \int_{-\infty}^{\infty} e^{iuv} \Psi_{\gamma}(v) \, dv \tag{C.4}$$

Martin (2007) obtains a close expression for $\Psi_{\gamma}(v)$, which is

$$\Psi_{\gamma}(v) = \frac{1}{2\pi} \frac{\Gamma(\gamma/2 - iv)\Gamma(\gamma/2 + iv)}{\Gamma(\gamma)} \tag{C.5}$$

Replacing in C.3, the right part of the equation becomes

$$= D_{it} e^{-\gamma/2(y_{1t}+y_{2t})} \mathbb{E}_t \left[ \int_{-\infty}^{\infty} e^{iv(y_{2t}-y_{1t}+\tilde{y}_{2(s-t)}-\tilde{y}_{1(s-t)})} e^{(\mathbb{1}_{\{i=1\}}-\gamma/2)\tilde{y}_{1(s-t)}+(\mathbb{1}_{\{i=2\}}-\gamma/2)\tilde{y}_{2(s-t)}} \Psi_{\gamma}(v) \, dv \right]$$

$$= D_{it}\, e^{-\gamma/2(y_{1t}+y_{2t})} \int_{-\infty}^{\infty} \mathbb{E} \left[ e^{(\mathbb{1}_{\{i=1\}}-\gamma/2-iv)\tilde{y}_{1(s-t)}+(\mathbb{1}_{\{i=2\}}-\gamma/2+iv)\tilde{y}_{2(s-t)}} \right] e^{iv(y_{2t}-y_{1t})} \Psi_{\gamma}(v) \, dv \tag{C.6}$$

Last equality stands because of the independent increments of $\tilde{y}_{is}$. Finally

$$\mathbb{E}_t\left[\frac{D_{is}}{(D_{1s}+D_{2s})^\gamma}\right] = D_{it}e^{-\gamma/2(y_{1t}+y_{2t})}\int_{-\infty}^{\infty} e^{(s-t)\mathbf{c}(\mathbb{1}_{\{i=1\}}-\gamma/2-iv,\mathbb{1}_{\{i=2\}}-\gamma/2+iv)}e^{iv(y_{2t}-y_{1t})}\Psi_\gamma(v)\,dv \tag{C.7}$$

We now replace in our last expression for $P_{it}$

$$\begin{aligned} P_{it} = &\int_t^\infty e^{-\delta(s-t)}\,(D_{1t}+D_{2t})^\gamma\,D_{it}\,e^{-\gamma/2(y_{1t}+y_{2t})} \\ &\int_{-\infty}^{\infty} e^{(s-t)\mathbf{c}(\mathbb{1}_{\{i=1\}}-\gamma/2-iv,\mathbb{1}_{\{i=2\}}-\gamma/2+iv)}e^{iv(y_{2t}-y_{1t})}\Psi_\gamma(v)\,dv\,ds &\text{(C.8)} \\ &= D_{it}e^{-\gamma/2(y_{1t}+y_{2t})}\,(D_{1t}+D_{2t})^\gamma \\ &\int_t^\infty e^{-\delta(s-t)}\int_{-\infty}^{\infty} e^{(s-t)c_i(v)}e^{iv(y_{2t}-y_{1t})}\Psi_\gamma(v)\,dv\,ds &\text{(C.9)} \end{aligned}$$

Where $c_i(v) := \mathbf{c}(\mathbb{1}_{\{i=1\}}-\gamma/2-iv, \mathbb{1}_{\{i=2\}}-\gamma/2+iv)$ for simplifying notations. For conditions over the parameters given by $\delta - c_i(v) > 0$ for $i = 1, 2$, the integrand above is absolutely integrable. Using Fubini's theorem

$$\begin{aligned} P_{it} &= D_{1t}\,e^{-\gamma/2(y_{1t}+y_{2t})}\,(D_{1t}+D_{2t})^\gamma\int_{-\infty}^{\infty} e^{iv(y_{2t}-y_{1t})}\left(\int_t^\infty e^{-(s-t)(\delta-c_i(v))}\,ds\right)\Psi_\gamma(v)\,dv \\ &= D_{1t}\,e^{-\gamma/2(y_{1t}+y_{2t})}\,(D_{1t}+D_{2t})^\gamma\int_{-\infty}^{\infty}\frac{e^{iv(y_{2t}-y_{1t})}}{\delta-c_i(v)}\Psi_\gamma(v)\,dv &\text{(C.10)} \end{aligned}$$

Replacing with the process $s_t = \frac{D_{it}}{D_{1t}+D_{2t}}$ finally proves Proposition 3.1.

## APPENDIX D. RETURNS IN EQUILIBRIUM

Procedure for obtaining expected asset returns is the same as in Martin (2007). First we proceed solving the expected capital gains, i.e. $\mathbb{E}_t[dP]$. We start with the expressions for prices obtained above:

$$P_{it} = (D_{1t} + D_{2t})^\gamma \int_{-\infty}^{\infty} h_i(v) e^{(\mathbb{1}_{\{i=1\}} - \gamma/2)\tilde{y}_{1t} + (\mathbb{1}_{\{i=2\}} - \gamma/2)\tilde{y}_{2t}} \, dv \tag{D.1}$$

where

$$h_i(v) = \frac{\Psi_\gamma(v)}{\rho - c_i(v)} \tag{D.2}$$

Using the Newton's generalized binomial theorem, we may expand $(D_{1t}+D_{2t})^\gamma$. It follows that

$$P_{it} = \sum_{n=0}^{\infty} \binom{\gamma}{n} \int_{-\infty}^{\infty} h_i(v) e^{w_{in}(v) y_t} \, dv \tag{D.3}$$

with

$$w_{in} = (\mathbb{1}_{\{i=1\}} - \gamma/2 + n - iv, \mathbb{1}_{\{i=1\}} - \gamma/2 - n + iv) \tag{D.4}$$

Let us call $X = e^{w_{in} y_t}$. Using Ito's differential formula applied to jump-diffusions we may obtain an expression for $\mathbb{E}_t[dX]$:

$$\mathbb{E}_t[dX] = \tag{D.5}$$

$$e^{w_{in} y_t} \mathbb{E}_t \left[ (w_{in}' \mu + w_{in}' \Sigma w_{in}) dt + w_{in}' A dZ + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{w_{in} g^{(k)}(z_k)} - 1) N^{(k)}(dt, dz_k) \right]$$

$$= e^{w_{in} y_t} \left( w_{in}' \mu + w_{in}' \Sigma w_{in} + \sum_{k=1}^{l} \int_{\mathbb{R}} (e^{w_{in} g^{(k)}(z_k)} - 1) \, \nu^{(k)}(dz_k) \right) dt \tag{D.6}$$

$$= e^{w_{in} y_t} \mathbf{c}(w_{in}) dt \tag{D.7}$$

Then $\mathbb{E}_t[dP_t]$ is given by

$$\mathbb{E}_t[dP_{it}] = \left( \sum_{n=0}^{\infty} \binom{\gamma}{n} \int_{-\infty}^{\infty} h_i(v) e^{w_{in}(v) y_t} \mathbf{c}(w_{in}) \, dv \right) dt \tag{D.8}$$

Replacing this result and expressions for prices (D.3) in (3.17), and after some algebraic manipulations, we get expression (3.18) for returns.

## APPENDIX E. RISKLESS RATE IN EQUILIBRIUM

Procedure for obtaining risk free rate can be found in Martin (2007). First we need to price a bond $B_T$ that pays a unit in period $T$. We use our state-price deflator (3.3) to discount the payment:

$$B_T = \mathbb{E}_t \left[ e^{-\delta(T-t)} \left( \frac{D_{1T} + D_{2T}}{D_{1s} + D_{2s}} \right)^{-\gamma} \right] \tag{E.1}$$

$$= e^{-\delta(T-t)} \left( D_{1T} + D_{2T} \right)^{\gamma} \mathbb{E}_t \left[ \frac{1}{\left( D_{1T} + D_{2T} \right)^{\gamma}} \right] \tag{E.2}$$

Conditional expectation is solved analogously as we did with prices. Bond's price is then given by

$$B_T = e^{-\delta(T-t)} \left( D_{1t} + D_{2t} \right)^{\gamma}$$
$$e^{-\gamma/2(y_{1t}+y_{2t})} \int_{-\infty}^{\infty} e^{(T-t)\mathbf{c}(-\gamma/2-iv,-\gamma/2+iv)} e^{iv(y_{2t}-y_{1t})} \Psi_\gamma(v)\, dv \tag{E.3}$$

The interest yield $R_{t\to T}$ of a bond $t$ that pays a unit in $T$ is defined by the relation $B_T = e^{-R_{t\to T}(T-t)}$. Isolating the interest yield and replacing with our state process $s_t = \frac{D_{1t}}{D_{1t}+D_{2t}}$, it follows that

$$R_{t\to T} = \delta - \frac{1}{T-t} \log \left( \frac{1}{\sqrt{s_t^{\gamma} (1-s_t)^{\gamma}}} \int_{-\infty}^{\infty} \left( \frac{1-s_t}{s_t} \right)^{iv} e^{(T-t)\mathbf{c}(-\gamma/2-iv,-\gamma/2+iv)} \Psi_\gamma(v)\, dv \right) \tag{E.4}$$

Instantaneous riskless rate $r_t = \lim_{T\to t} R_{t\to T}$ is obtained using L'Hopital rule:

$$r_t = \delta - \frac{\int_{-\infty}^{\infty} \left( \frac{1-s_t}{s_t} \right)^{iv} \mathbf{c}(-\gamma/2 - iv, -\gamma/2 + iv) \Psi_\gamma(v)\, dv}{\int_{-\infty}^{\infty} \left( \frac{1-s_t}{s_t} \right)^{iv} \Psi_\gamma(v)\, dv} \tag{E.5}$$

By definition $(2\cosh(u/2))^{\gamma} \int_{-\infty}^{\infty} e^{iuv} \Psi_\gamma(v)\, dv = 1$. Using this fact and regrouping terms we finally obtain expression (3.16).

## APPENDIX F. AGENT'S UTILITY FUNCTION

The agent's $i$ utility function before $\tau$ is given by

$$V^i(D_{1t}, D_{2t}) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\delta(s-t)} u(C_{is})\, ds\right] \tag{F.1}$$

$$= \mathbb{E}_t\left[\int_t^{\tau} e^{-\delta(s-t)} u(D_{is})\, ds + e^{-\delta(\tau-t)} \int_\tau^{\infty} e^{-\delta(s-\tau)} u(C_{is})\, ds\right] \tag{F.2}$$

For simplifying the resolution, we will use the utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, and then subtract the missing constant $1/(1-\gamma)$. After rearranging some terms

$$\begin{aligned} &e^{-\delta t} V^i\left(D_{1t}, D_{2t}\right) + \mathbb{E}_t\left[\int_0^t e^{-\delta s} u(D_{is})\, ds\right] \\ &= \mathbb{E}_t\left[\int_0^{\tau} e^{-\delta s} u(D_{is})\, ds + e^{-\delta \tau} \int_\tau^{\infty} e^{-\delta(s-\tau)} u(C_{is})\, ds\right] \end{aligned} \tag{F.3}$$

We see that the right part of this equation is a martingale in $\mathcal{F}_t$, so for any time $t < \tau$ we must have that $\mathbb{E}_t\left[d\left(e^{-\delta t} V^i\right) + e^{-\delta t} u(D_{it})\right] = 0$[1]. Using Ito's differential formula (no jumps):

$$\begin{aligned} &d\left(e^{-\delta t} V^i\right) = e^{-\delta t}\Bigg(-\delta V^i dt + \\ &V_x^i dD_{1t} + V_y^i dD_{2t} + \frac{1}{2} V_{xx}^i d[D_{1t}, D_{1t}] + V_{xy}^i d[D_{1t}, D_{2t}] + \frac{1}{2} V_{yy}^i d[D_{2t}, D_{2t}]\Bigg) \end{aligned} \tag{F.4}$$

Imposing the previous martingale condition, and replacing with the processes, we get the following PDE

$$-\delta V^1 + \mu_{D_1} x V_x^1 + \mu_{D_2} y V_y^1 + \frac{1}{2} x^2 V_{xx}^1 A_1 A_1^T + \frac{1}{2} y^2 V_{yy}^1 A_2 A_2^T + xy V_{xy}^1 A_1 A_2^T + u(x) = 0 \tag{F.5}$$

$$-\delta V^2 + \mu_{D_1} x V_x^2 + \mu_{D_2} y V_y^2 + \frac{1}{2} x^2 V_{xx}^2 A_1 A_1^T + \frac{1}{2} y^2 V_{yy}^2 A_2 A_2^T + xy V_{xy}^2 A_1 A_2^T + u(y) = 0 \tag{F.6}$$

[1] We shall permit ourselves this abuse of notation

These are linear second order PDE. Verifying the positivity of $(xyA_1A_2^T)^2 - 4\frac{1}{2}x^2A_1A_1^T\frac{1}{2}y^2A_2A_2^T$, we see the the equation parabolic for $\rho^2 = 1$ and elliptic in other cases. Border conditions are given by replacing the IPO trigger condition in (4.1): For $D_{1\tau}$ and $D_{2\tau}$ such that $\frac{D_{1\tau}}{D1\tau + D_{2\tau}} = s_\tau$

$$V^i\left(D_{1\tau}, D_{2\tau}\right) = \mathbb{E}_t\left[\int_\tau^\infty e^{-\delta(s-\tau)}u(C_{is})\,ds\right] \tag{F.7}$$

In a similar way as we did in (A.2), this expression can be written like

$$V^i\left(D_{1\tau}, \frac{1-s_\tau}{s_\tau}D_{1\tau}\right) = \frac{1}{1-\gamma}\left(\frac{W_{i\tau}}{C_{i\tau}^\gamma} - \frac{1}{\delta}\right) \tag{F.8}$$

And finally, because of (3.14), (3.6) and that $C_{1t} = \omega_\tau\left(D_{1t} + D_{2t}\right)(1-\alpha)$, we have that for any $t \geq \tau$

$$\frac{W_{it}}{C_{1t}^\gamma} = \left(\omega_{i\tau}\left(1-\alpha\right)\right)^{1-\gamma} D_{1t}\left(D_{1t}D_{2t}\right)^{-\gamma/2}\int_{-\infty}^\infty \left(\frac{1-s_t}{s_t}\right)^{iv}\left(\frac{1}{\delta - c_1(v)} + \frac{(1-s_t)/s_t}{\delta - c_2(v)}\right)\Psi_\gamma(v)\,dv \tag{F.9}$$

where $\omega_{1\tau} = \omega_\tau$ and $\omega_{2\tau} = 1 - \omega_\tau$.

Just as we did in (A.10), we will consider a solution of the form $V^i\left(x, y\right) = x(xy)^{-\gamma/2}h^i(u)$ with $u = \ln\left(\frac{1-s}{s}\right) = \ln\left(\frac{y}{x}\right)$. This solution will then satisfy

$$x(xy)^{-\gamma/2}\left(\theta_0 h^1(u) + \theta_1 h^{1'}(u) + \theta_2 h^{1''}(u) + \frac{e^{\gamma/2u}}{1-\gamma}\right) = 0 \tag{F.10}$$

$$x(xy)^{-\gamma/2}\left(\theta_0 h^2(u) + \theta_1 h^{2'}(u) + \theta_2 h^{2''}(u) + \frac{e^{1-\gamma/2u}}{1-\gamma}\right) = 0 \tag{F.11}$$

(F.12)

With

$$
\begin{aligned}
\theta_0 &= -\delta + (1-\gamma/2)\mu_{D_1} - \gamma/2\mu_{D_2} + 1/2(\gamma^2/4 - \gamma/2)A_1A_1^T \\
&\quad + (\gamma^2/4 - \gamma/2)A_1A_2^T + 1/2(\gamma^2/4 + \gamma/2)A_2A_2^T \qquad \text{(F.13)} \\
\theta_1 &= \mu_{D_2} - \mu_{D_1} + 1/2(\gamma - 1)A_1A_1^T + A_1A_2^T + 1/2(-\gamma - 1)A_2A_2^T \\
\theta_2 &= 1/2A_1A_1^T - A_1A_2^T + 1/2A_2A_2^T
\end{aligned}
$$

The homogeneous solution to $h^i$ is given by

$$h_H^i(u) = C_i e^{\lambda u} \tag{F.14}$$

Where $\lambda$ satisfies $\theta_0 + \theta_1\lambda + \theta_2\lambda^2 = 0$. The particular solution to $h^i$ is given by

$$h_P^1(u) = A_1 e^{\gamma/2u} \tag{F.15}$$

$$h_P^2(u) = A_2 e^{1-\gamma/2u} \tag{F.16}$$

Where

$$A_1 = -\frac{\frac{1}{1-\gamma}}{(\gamma/2)^2\theta_2 + \gamma/2\theta_1 + \theta_0} = \frac{1}{(1-\gamma)(c(1-\gamma,0))} \tag{F.17}$$

$$A_2 = -\frac{\frac{1}{1-\gamma}}{(1-\gamma/2)^2\theta_2 + (1-\gamma/2)\theta_1 + \theta_0} = \frac{1}{(1-\gamma)(c(0,1-\gamma))} \tag{F.18}$$

Finally, adding the constant we removed in (4.1), we get that the agent's utility function before $\tau$ is given by

$$V^1(x,y) = x(xy)^{\gamma/2}\left(C_1e^{\lambda u} + A_1e^{\gamma/2u}\right) - \frac{1}{(1-\gamma)\delta} \tag{F.19}$$

$$V^2(x,y) = x(xy)^{\gamma/2}\left(C_2e^{\lambda u} + A_2e^{1-\gamma/2u}\right) - \frac{1}{(1-\gamma)\delta} \tag{F.20}$$

The constant $C_i$ is fixed so that the solution satisfies the terminal condition (F.8)

For $t > \tau$, the utility function of each agent is given by

$$\frac{1}{1-\gamma}\left(\frac{W_{it}}{C_{it}^{\gamma}} - \frac{1}{\delta}\right) \tag{F.21}$$

Where the fraction $\frac{W_{it}}{C_{it}^{\gamma}}$ is the same as in (F.9).