(2 + 1)-Dimensional Charged Black Hole in Topologically Massive Electrodynamics

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The 2 + 1 black hole coupled to a Maxwell field can be charged in two different ways. Besides a Coulomb field, whose potential grows logarithmically in the radial coordinate, there also exists a topological charge due to the existence of a noncontractible cycle. The topological charge does not gravitate and is somehow decoupled from the black hole. This situation changes if one turns on the Chern-Simons term for the Maxwell field. First, the flux integral at infinity becomes equal to the topological charge. Second, demanding regularity of the black hole horizon, the Coulomb charge must vanish identically. Hence, in 2 + 1 topologically massive electrodynamics coupled to gravity, the black hole can support holonomies only for the Maxwell field. This means that the charged black hole is constructed from the vacuum by means of spacetime identifications.

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The consistency of general relativity with quantum mechanics is still one the most important problems of theoretical physics. Since the problem was first formulated several decades ago, it became clear that black holes would play a key role as a tool to explore quantum gravity. Two striking properties of these particular solutions of Einstein equations—which have driven most of the work on black hole quantum mechanics for many years—are Hawking radiation and the so called "no hair" theorems.

The no hair theorems (see [1] for a review) imply that black holes are described only by their mass, angular momentum, and charge. In string theory, for example, charged objects have played a key role in recent developments. D-branes and most extended objects are supported by p-form fields carrying some conserved charge. Black holes on these branes are also known, and, in fact, they have been the first examples for which an statistical description of the Bekenstein-Hawking entropy is available [2].

Black holes [3] also exist in the simpler setting of three-dimensional gravity [4], and they share most of the properties of the higher dimensional ones. Even though 2 + 1 gravity does not contain gravitational waves, it is now clear that it encodes a number of interesting properties [5]. To quote some examples, we mention the discovery by Brown and Henneaux [6] of a centrally extended asymptotic conformal algebra, and its Chern-Simons formulation [7].

In this Letter, we are interested in the charged version of the Banados-Teitelboim-Zanelli (BTZ) [3] black hole. In 2 + 1 dimensions, the action describing the Maxwell field can be generalized to contain a Chern-Simons ("topological mass") term [8]. We consider then the action

$$I = \int \sqrt{-g}\left(R + 2 - \frac{\kappa}{4} F^{\mu\nu} F_{\mu\nu}\right) - \frac{\alpha}{2} \int \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}. \quad (1)$$

We stress that both the Maxwell and the Chern-Simons terms are quadratic in A and gauge invariant. There is thus no *a priori* reason to exclude one in favor of the other. Furthermore, the "massive" character of the Chern-Simons term implies that the Brown-Henneaux [6] symmetry still applies in this theory, as opposed to pure Maxwell theory.

In 5 and 11 dimensions, Chern-Simons terms arise as a consequence of supersymmetry. Black hole solutions for the five-dimensional version of (1) have been found recently in [9]. As we shall see, in three dimension there are some surprises.

Particular black hole solutions to (1) are known in various cases. For $\kappa = \alpha = 0$, one finds the uncharged BTZ black hole [3],

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2\left(d\varphi + \frac{J}{2r^2}dt\right)^2, \quad (2)$$

where

$$N^2(r) = -M + r^2 + \frac{J^2}{4r^2}. \quad (3)$$

The asymptotic charges for this solution are M and J. The geometry is regular for all positive values of r, and there is a regular event horizon provided $M > |J|$.

If $\kappa \neq 0$ but still $\alpha = 0$, one finds a more complicated set of equations whose solution has been found in [10] (this system was not fully solved in [3]). The black hole is now described by three asymptotic conserved charges M, J, and the charge C. If $J = 0$, the charged metric has the form (2) with $N^2(r) = -M + r^2 + C^2 \log(r)$.

If $\kappa = 0$ and $\alpha \neq 0$, the Abelian field decouples from gravity. Globally, however, the gauge field can feel the presence of the black hole by developing a nonzero holonomy around the noncontractible loop. The general solution in this case is the vacuum black hole (2) supplemented with the constant value for the gauge field,

$$A_\varphi = \frac{Q}{2\pi} \Rightarrow \oint A = Q. \tag{4}$$

The "topological charge" Q cannot be eliminated by a well-defined gauge transformation because φ is compact [11]. Furthermore, for $\kappa = 0$ and $\alpha \neq 0$, Q is equal to the electric charge of the system, defined as the conserved quantity associated with nontrivial asymptotic gauge transformations. It can also be represented as the zero mode of the infinite-dimensional asymptotic $U(1)$ Kac-Moody algebra. We shall prove in this Letter that this configuration is stable against the incorporation of the Maxwell term that is turning on the coupling κ.

Consider then the case $\kappa \neq 0$ and $\alpha \neq 0$. In what follows, we set $\kappa = 1$. Self-dual [13], particlelike solutions to this system were first discussed in [14] and further analyzed in [15]. Our interest will be focused on black holes solutions.

We start by writing the equations of motion. For our purposes it will be convenient to use the following parametrization for the black hole ansatz:

$$ds^2 = \frac{dr^2}{h^2 - pq} + pdt^2 + 2hdtd\varphi + qd\varphi^2, \tag{5}$$

where p, q, h are functions of r only. For the gauge field, we write

$$A_t(r) = \Phi(r), \tag{6}$$

$$A_\varphi(r) = \frac{Q}{2\pi} + \chi(r). \tag{7}$$

We have isolated the constant part of A_φ (holonomy), and assume that both $\Phi(r)$ and $\chi(r)$ vanish at infinity (this is true only for $\alpha \neq 0$).

Inserting (5)–(7) into Einstein equations coupled to the Maxwell field and taking appropriate combinations, we find the remarkable simple set of equations (prime indicates radial derivative)

$$h'' = -\Phi'\chi', \tag{8}$$

$$p'' = -\Phi'^2, \tag{9}$$

$$q'' = -\chi'^2. \tag{10}$$

The equations also imply $(h^2 - pq)'' = h'^2 - p'q' + 4$ (recall that we are using $\Lambda = -1$). It is direct to see, however, that (8)–(10) imply $(h^2 - pq)''' = (h'^2 - p'q')'$, and hence we can omit this extra equation provided we fix the integration constant as 4.

The Maxwell-Chern-Simons equations become

$$h\Phi' - p\chi' = 2\alpha\Phi, \tag{11}$$

$$q\Phi' - h\chi' = 2\alpha\chi. \tag{12}$$

These equations are first integrals of the original ones. The two integration constants are the holonomy, $Q/2\pi$, represented by the constant part of A_φ, and a constant added to A_t which is trivial and can be gauged away.

The reduced set of Eqs. (8)–(12) admits a Lagrangian representation [14],

$$L = \frac{1}{4}\,\mathrm{Tr}(\Omega'^2) + \alpha\bar{A}A' - 2\alpha^2\bar{A}\Omega^{-1}A, \tag{13}$$

where the functions h, p, q and Φ, χ are collected in the $SL(2, \Re)$ matrix Ω and "spinor" A,

$$\Omega = \begin{pmatrix} h & -p \\ q & -h \end{pmatrix}, \qquad A = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}. \tag{14}$$

The "Dirac conjugate" is defined as $\bar{A} = A^t i = (\chi, -\Phi)$ where i is the real antisymmetric matrix in two dimensions. Equations (8)–(10) and (11) and (12) take the form $\Omega'' = -A' \otimes \bar{A}'$ and $\Omega A' = 2\alpha A$, respectively, and can be derived by extremizing (13).

The Lagrangian (13) is invariant under $SL(2, \mathrm{Re})$ rotations [this symmetry is implicit in the ansatz (5)]

$$\Omega \rightarrow U^{-1}\Omega U, \qquad A \rightarrow U^{-1}A, \qquad \bar{A} \rightarrow \bar{A}U, \tag{15}$$

where U is a *constant* matrix with $\det(U) = 1$. The corresponding Noether charge, satisfying $K' = 0$, is given by

$$K = \frac{1}{2}[\Omega, \Omega'] + \alpha A \otimes \bar{A}. \tag{16}$$

The Lagrangian (13) is also invariant under constant translations in the radial coordinate $r \rightarrow r + a$ and the associated Noether charge is the "energy,"

$$\mathcal{E} = \frac{1}{4}\,\mathrm{Tr}(\Omega'^2) + 2\alpha^2\bar{A}\Omega^{-1}A. \tag{17}$$

This integral is not, however, an arbitrary constant, but it is fixed by Einstein equations. In fact, it is direct to prove that $2\mathcal{E} = (h^2 - pq)'' - (h'^2 - p'q')$. Hence, according to the discussion after Eq. (10), we must fix $\mathcal{E} = 2$.

The vacuum solution (BTZ black hole) in this representation is given by $\Phi = 0 = \chi$, and

$$h(r) = J, \qquad p(r) = 4(M - r), \qquad q(r) = r. \tag{18}$$

In fact, replacing (18) in (5) and making the radial redefinition $r \rightarrow r^2$, one obtains (2).

We would like to add charge to (18). We first study the asymptotic structure of the charged solution. Setting (18) as the background, we develop a series expansion in powers of $1/r$ for the solution. This can be done to any desired order. We display here the first few terms,

$$h = J + \frac{C^2\alpha}{4(2\alpha+1)}\frac{1}{r^{2\alpha}}\left(1 + \frac{M\alpha}{r} + \cdots\right),$$
$$p = 4(M-r) - \frac{C^2\alpha}{2(2\alpha+1)}\frac{1}{r^{2\alpha}}\left(1 + \frac{2M\alpha - J}{2\alpha+1}\frac{\alpha}{r} + \cdots\right),$$
$$q = r - \frac{C^2\alpha}{8(2\alpha+1)}\frac{1}{r^{2\alpha}}\left(1 + \frac{2M\alpha + 2M + J}{2\alpha+1}\frac{\alpha}{r} + \cdots\right),$$
$$\Phi = \frac{C}{r^\alpha}\left(1 + \frac{2M\alpha - J}{2(2\alpha+1)}\frac{\alpha}{r} + \cdots\right),$$
$$\chi = -\frac{C}{2r^\alpha}\left(1 + \frac{(J + 2M\alpha + 2M)}{2(2\alpha+1)}\frac{\alpha}{r} + \cdots\right).$$

Here, C is an integration constant, which will be called "Coulomb charge."

There exists another solution whose gauge field diverges as r^α, and hence we discard it. We assume $\alpha > 0$. Note that the $C\log(r)$ structure arising in pure Maxwell theory has been replaced by C/r^α. This is a consequence of the massive character of the Chern-Simons term.

The asymptotic solution is thus characterized by four parameters, the mass M, angular momentum J, topological charge Q, and Coulomb charge C. Let us prove that the associated electric charge is equal to Q.

In the presence of the Chern-Simons term, the definition of electric charge has some subtleties. This problem was first studied in [16]. See [17] for recent discussions. For the case at hand, the electric charge of the system is [16]

$$\int_\gamma ({}^*F + 2\alpha A), \tag{19}$$

where the integral encloses the origin $r = 0$ and it is assumed to be outside sources. As usual, the value of the integral does not depend locally on γ. We choose the curve to be a circle of a large radius at some constant time. From the above ansatz, we find ${}^*F_\phi = -q\Phi' + h\chi'$, and hence the charge becomes

$$\int d\phi\left[-q\Phi' + h\chi' + 2\alpha\left(\frac{Q}{2\pi} + \chi\right)\right] = 2\alpha Q, \tag{20}$$

where we have used the equation of motion (12). We thus find that the electric charge of the system is proportional to the topological charge Q. The Coulomb charge C is not related to an asymptotic symmetry, and it would appear as "hair" in the corresponding black hole solution.

We now turn to the problem of finding black hole solutions to (8)–(12), satisfying the asymptotic conditions found above. To analyze the horizon geometry, we rewrite the ansatz (5) in its "Arnowitt-Deser-Misner" form

$$ds^2 = -\frac{h^2 - pq}{q}dt^2 + \frac{dr^2}{h^2 - pq} + q\left(d\varphi + \frac{h}{q}dt\right)^2. \tag{21}$$

From here, we see that the structure of horizons is controlled by the function

$$f(r) \equiv h^2 - pq. \tag{22}$$

A point $r = r_+$ satisfying the following three conditions: (i) $f_+ = 0$, (ii) the horizon area q_+ is not zero, and (iii) the angular velocity of the horizon $\frac{h_+}{q_+}$ is not singular, defines a regular event horizon. Here and below, a subscript + means the corresponding function evaluated at the horizon.

Although the system (8)–(12) does not look too complicated, its solution for all values of r has escaped us. (See [14] for a particular solution describing "particles.") Nevertheless, without knowledge of the exact solution, we shall be able to prove that a regular horizon can exist if and only if the Coulomb charge vanishes identically,

$$C = 0. \tag{23}$$

To this end, we first prove that the function q must be positive for all $r > r_+$. Since q_+ is the horizon area, we have $q_+ > 0$. From (10) we observe that q'' is negative for all r. If $q'_+ < 0$, then $q(r)$ would eventually become negative, in contradiction with its asymptotic value given above. Thus, q'_+ must be positive and, as a consequence, q is positive for all $r > r_+$.

In a similar way, we now prove that p must be negative for all $r > r_+$. First, we note that one can always choose the angular velocity at the horizon to be zero, that is, $h_+ = 0$. This is achieved by the transformation $\varphi \to \varphi + wt$, which, acting on (21), has the effect of shifting $h/q \to h/q + w$ [18]. From now on, we work on this frame. Next, we note that since $h_+^2 - p_+q_+ = 0$ and q_+ is finite, it follows that $p_+ = 0$. This implies that

$$f'_+ = 2h_+h'_+ - p'_+q_+ - p_+q'_+ = -p'_+q_+. \tag{24}$$

The function f is positive for $r > r_+$ and vanishes at $r = r_+$. This implies that $f'_+ \geq 0$, and since $q_+ > 0$, we find that p'_+ must be negative or zero. From Eq. (9) we see that p'' is negative for all r. We conclude that p' must be negative all the way from r_+ to infinity. Since p vanishes at the horizon, this means that p must be negative for all $r > r_+$.

We now analyze the behavior of the gauge field. First, we note that $\Phi_+ = 0$. This follows directly from Eq. (11), with $\alpha \neq 0$. On the other hand, from the asymptotic series displayed above, we see that both functions Φ and χ vanish as $r \to \infty$. Then, the function $\sigma \equiv \Phi\chi$ vanishes both at the horizon and at infinity. (We assume here that χ_+ is finite, which is required for regularity of the horizon [19].) Since σ must be continuous, we find that either σ is zero everywhere or has an extremum at some value $r = r_0$. If the latter is true, then the derivative of σ at each side of r_0 has different sign, and therefore there must be a region where σ' is negative. But this is a contradiction to the equation of motion. In fact, multiplying Eq. (15) by χ' and Eq. (16) by Φ', and adding them together, one obtains

$$q(\Phi')^2 - p(\chi')^2 = 2\alpha\sigma'. \tag{25}$$

The left-hand side of this expression is non-negative, because $p < 0$ and $q > 0$ for all $r > r_+$. Therefore σ' cannot take negative values. We conclude that σ must be identically zero everywhere, and thus either Φ or χ must be zero. By inspection of the asymptotic solution, this imply that C must vanish.

To summarize, we have shown in this Letter that BTZ black holes coupled to Maxwell-Chern-Simons electrodynamics can support holonomies for the gauge field, but not for a local electromagnetic field. The charged black hole thus reduces to the BTZ metric plus the holonomy (4). A star could support a local field characterized by C, and we have found its explicit asymptotic form. Upon gravitational collapse, however, the field must be expelled before a regular horizon could be formed. We hope to come back to this problem in the future. A more detailed analysis of the results presented here will be presented elsewhere.

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[19] In a gauge theory, one should impose regularity only on the field strengths, not on the gauge potentials. However, in this case the conserved quantity K again implies that χ_+ must be regular at the horizon for the geometry to be well defined.